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ELECTROMAGNETIC WAVE PROPAGATION IN NONLINEAR LAYERED WAVEGUIDE STRUCTURES

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PREFACE

In this monograph, modern mathematical methods for investigation of polarized electromagnetic wave propagation in nonlinear waveguide structures are considered. These methods have been developed by the authors over the last several years. The structures under consideration are layers and circle cylindrical waveguides filled with nonlinear medium. The permittivity of the medium is a function of electric field intensity.

The problems can be reduced to nonlinear boundary eigenvalue problems for ordinary differential equations. In particular, we study eigenvalue problems as mathematical problems for Maxwell equations. Although the attempts to solve such problems had been known for a long period the main results have been obtained recently. For over 35 years many papers were published in scientific journals and different approaches were suggested. As it seems to the authors they developed the general approach for such problems. It is called *Integral Dispersion Equation Method*. This method allows to study problems of wave propagation from the common standpoint and allows to obtain information about the behavior of dispersion curves. It should be noticed that obtained dispersion equations can be studied both analytical and numerical methods. Propagation constants can be calculated from dispersion equations.

We would like to emphasize that suggested approach allows to study both nonlinear materials and nonlinear metamaterials.

Monograph consists of introduction and two parts, the first of which is devoted to investigation of boundary eigenvalue problems in layers and the second one is devoted to boundary eigenvalue problems for circle cylindrical waveguide. Each part contains several chapters, and each chapter contains several sections. Such a structure of the monograph allows to read every section independently of the others. In each section we assume continuous numbering of formulas. If we refer to a formula from another section of a given chapter, we add the number of the section before the number of the formula. Similarly we add the number of the chapter if we refer to a formula from another chapter. We assume similar numbering for definitions, theorems, lemmas and so on.

This monograph can be used by scientific researchers, post-graduate and graduate students who investigate electromagnetic problems as well as mathematical models of processes of electromagnetic wave propagation.

We hope that the study of the methods considered in this monograph extends the mathematical erudition of the reader in the area of electrodynamics, and allows the investigation of new complicated electromagnetic problems.

Yury Smirnov, Dmitry Valovik

INTRODUCTION

Problems of electromagnetic wave propagation in nonlinear waveguide structures are intensively investigated during several decades. First known studies about the problems are the monographs [3, 9]. Propagation of electromagnetic wave in layers and circle cylindrical waveguides are among such problems. Phenomena of electromagnetic wave propagation in nonlinear media have original importance[1] and also find a lot of applications, for example, in plasma physics, microelectronics, optics, laser technology [3, 9, 35]. Nonlinear effects can be observed in liquid crystals [26], semiconductors (such as InSb, HgCdTe) etc. Developing of mathematical models and methods for their solving for such problems is getting more and more important.

There are a lot of different nonlinear phenomena in media when electromagnetic wave propagates, such as self-focusing, defocusing, and self-channeling etc. [9, 35]. In order to describe the influence of different phenomena it is necessary to develop new analytical and numerical methods for studying such problems.

Investigation of nonlinear phenomena leads us to solve nonlinear differential equations. More precisely, it is necessary to solve nonlinear boundary eigenvalue problems, which rarely can be solved analytically. One of the important nonlinear phenomenon is the case when the permittivity of the sample depends on electric field intensity. (For the case of Kerr nonlinearity[2] see, for example, [17, 18]). Solving such problems in a strong electromagnetic statement is very difficult. Many researchers try to simplify these problems in different ways [12, 44].

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[1] Such phenomena in some cases can be described by nonlinear boundary eigenvalue problems for the Maxwell equations, which are not easy to solve.

[2] When one says that the permittivity ε is described by Kerr law this means that $\varepsilon = \varepsilon_{\text{const}} + \alpha|\mathbf{E}|^2$, $\varepsilon_{\text{const}}$ is the constant part of the permittivity ε; α is the nonlinearity coefficient; $|\mathbf{E}|^2 = E_x^2 + E_y^2 + E_z^2$ is the squared absolute value of the electric field $\mathbf{E} = (E_x, E_y, E_z)$.

Perhaps, the paper [17] was the first study where some problems of electromagnetic wave propagation are considered in a strong electromagnetic statement. Propagation of polarized electromagnetic waves in a layer and in a circle cylindrical waveguide with Kerr nonlinearity is considered in this paper.

Problems of electromagnetic wave propagation in a linear layer (with constant permittivity) and in a linear circle cylindrical waveguide were deeply studied many years ago, see, for example [1, 64, 68]. Such problems are formulated as boundary eigenvalue problems for ordinary differential equations. However, in the nonlinear cases many researches (see, for example, [23, 32, 33, 46, 52]) pay more attention to solve the differential equations and do not point out to find dispersion equations[2] (DE). In the most cases the equations can not be integrated in an explicit form. Of course, if one has the explicit solutions of the differential equations it is easy to write the DE. Therefore, when the equations can not be integrated things do not go to a DE. However, in some cases the DE can be found in an explicit form and it is not necessary to have explicit solutions of differential equations. We should underline that the problems under consideration are exactly boundary eigenvalue problems. Indeed, the main interest in these problems is the value of the spectral parameter (eigenvalues), which corresponds to the propagating wave. If an eigenvalue is known it is easy to solve differential equations numerically. Otherwise numerical methods can not be successfully applied.

Let us discuss in detail the case of Kerr nonlinearity. Propagation of TE waves were most studied. The work [45] is devoted to the problem of electromagnetic wave propagation in a nonlinear dielectric layer with absorption and the case of Kerr nonlinearity is considered separately. Propagation and reflection of TE wave in a nonlinear layer are studied in the article [46]. In this case the equations are integrated in the term of Weierstrass elliptic function. One

[2]From the mathematical standpoint the DE is an equation with respect to spectral parameter. Analysis of this equation allows us to make conclusions about problem's solvability, eigenvalues' localization etc.

of the most interesting results about propagation of TE waves in a layered structure with Kerr nonlinearity is the paper [48].

The case of TM wave propagation in a nonlinear medium is more complicated. This is due to the fact that two components of the electric field complicate the analysis [11]. As it is known the permittivity is expressed in terms of the electric field components and two components lead to more complicated dependence of the permittivity on the electric field intensity. Hence we obtain more complicated differential equations.

In the work [12] a linear dielectric layer is considered. The layer is located between two half-spaces, the half-spaces are filled by nonlinear medium with Kerr nonlinearity. This problem for TE waves is solved analytically [10, 50]. For the TM case in [12] the DE is obtained, which is an algebraic equation. It should be noticed that in [12] authors simplify the problem. Earlier in [2] with other simplifying assumption the DE is obtained (authors take into account only one component E_x of the electric field). Later in [51] it is proved that the dominating nonlinear contribution in the permittivity is proportional to the transversal component E_z. In the works [23, 69] propagation of TM waves in a nonlinear half-space with Kerr nonlinearity is considered. Formal solutions of differential equations in quadratures are obtained. In the paper [23] DEs for isotropic and anisotropic media in a half-space with nonlinear permittivity are shown. The DEs are rational functions with respect to the value of field's components at the interface. Authors found the first integral of the system of differential equations (so called a conservation law). This is also very interesting work to study.

In the case of TE wave you can see the papers [24, 25, 31, 66]. Propagation of TM wave in terms of the magnetic component are studied in [32, 33]. The paper [2] is devoted to the question (from physical standpoint) why it is possible to take into account only one component of the electric field in the expression for permittivity in the case of TM waves in a nonlinear layer. The results are compared with the case of TE waves.

The most important results about TM wave propagation in a layer with Kerr nonlinearity (system of differential equations, first integral) and a circle cylindrical waveguide (system of differential equations) were obtained in [17, 18] (1971–1972 years). In some papers (for example, [32]) polarized wave propagation in a layer with arbitrary nonlinearity is considered. However, DEs were not obtained and no results about solvability of the boundary eigenvalue problem were obtained as well. The problem of TM wave propagation in a layer with Kerr nonlinearity is solved at first for a thin layer [80] and then for a layer of arbitrary thickness [77, 79, 63]. Theorems of existence and localization of eigenvalues are proved in [72]. Some numerical results are shown in [74, 75].

In this monograph the DE is an equation with additioanl conditions. Only for linear media (the permittivity is a constant) in a layer or in a circle cylindrical waveguide the DEs are sufficiently simple (but even for these cases the DEs are transcendental equations). For nonlinear layers the DEs are rather complicated combinations of nonlinear integral equations, where integrands are defined by implicit algebraic (or transcendental) functions. For the cases of nonlinear circle cylindrical waveguides some results were obtained only for Kerr nonlinearity. It should be stressed that in spite of the fact that DEs are complicated they can be rather easily solved numerically.

These DEs allow to study both nonlinear materials and nonlinear metamaterials. It should be noticed that in the monograph materials with arbitrary permittivity and constant positive permeability are studied. But it is not difficult to take into account the sign of the permeability. In other words, represented DEs allow to study rather broad spectrum of materials.

The method of obtaining DEs is called *Integral Dispersion Equations Method*. For layers this method allows to study wide class of nonlinearities for the TM wave and arbitrary nonlinearities for the TE wave. On the basis of the DEs for linear layers we study linear metamaterials.

We assume that a permittivity is a diagonal tensor. The tensor is represented as a diagonal 3×3 matrix. The permittivity is a tensor with different components for anisotropic media (for isotropic media the permittivity is a scalar). However, for TE waves even for anisotropic media only one component of this diagonal tensor is taken into account. For TM waves we consider both isotropic and anisotropic media.

When we speak about nonlinear boundary eigenvalue problems we mean that differential equations and boundary conditions nonlinearly depend on the spectral parameter and also the differential equations nonlinearly depend on the unknown functions. All these facts do not allow to apply well-known methods of investigation of spectral problems.

Problems of propagation of TE and TM waves in nonlinear circle cylindrical waveguides are also considered. These problems are more complicated in comparison with corresponding problems in nonlinear layers. And even in the case of Kerr nonlinearity the results are not so complete as in the case in layers.

Let us give a survey of the monograph.

In Chapter 1 the results on electromagnetic wave propagation in a layer with constant permittivity are laid down. There are a lot of works where the issue is discussed [1, 16, 19, 40, 64]. Perhaps we do it in the way, which is more suitable for our goals.

In Chapter 2 we consider TE wave propagation in a linear layer. The dispersion equation is derived and analyzed. It is proved that there are no TE waves in a linear metamaterial layer. Some numerical results are shown. The following works [1, 16, 19, 40, 64] were very useful for us.

In Chapter 3 it is considered TE wave propagation in a layer with arbitrary nonlinearity. The permittivity in the layer is an arbitrary function with respect to electric field intensity. We derive the DE of the problem. Using this equation it is possible to study wide kind of nonlinearity. The problem statement is presented in [17, 18]. In this chapter numerical results for more complicated nonlinearity than Kerr-type one are shown. The content of the chapter is based on [70].

In Chapter 4 we apply the general technique (developed in Chapter 3) to the TE wave propagation in a layer with generalized Kerr nonlinearity. The DE is derived and discussed. The similar problem for Kerr nonlinearity is studied by using elliptic functions in [73].

In Chapter 5 we discuss TM wave propagation in a layer with constant permittivity. The permittivity is described by a diagonal tensor. It turns out that it is possible to consider the layer filled with metamaterial. The numerical results are shown. Some results of the chapter were published in [76].

Chapter 6 is devoted to TM wave propagation in a layer with arbitrary nonlinearity. The permittivity in the layer is an arbitrary function with respect to electric field intensity. As a matter of fact, the permittivity is not quite arbitrary function. There is a condition that the function obeys (details see in the chapter). We derive the DE of the problem. The problem statement is presented in [17, 18]. The content of the chapter is based on [71].

In Chapter 7 we apply the general technique (developed in Chapter 6) to the TM wave propagation in a isotropic layer with Kerr nonlinearity. The DE is derived and discussed. It is shown that passage to the limit (when the nonlinearity coefficient tends to zero) gives a linear case discussed in Chapter 5. The first approximation for eigenvalues of the problem is obtained. Some numerical results are also presented. The results of the chapter were published in [68, 71, 74, 76–79].

In Chapter 8 we apply the general technique (developed in Chapter 6) to the TM wave propagation in a anisotropic layer with Kerr nonlinearity. The DE is derived and discussed. The results of the chapter is based on [74]. In this paper some numerical results are also presented.

Chapter 9 is devoted to well-known results about electromagnetic wave propagation in circle cylindrical waveguides. It is known that it is possible to study TE and TM waves instead of general electromagnetic field. This approach allows to pass from partial differential equations (PDE) to ordinary differential equations (ODE). There are a lot of works where the issue is discussed [1, 16, 19, 40, 64]. Perhaps we do it in the way, which is more suitable for our goals. Also we discuss the existence of nonlinear waves. It appears that there are restrictions for existence of nonlinear waves in such waveguide structure (for details see the chapter).

In Chapter 10 we discuss TE wave propagation in a circle cylindrical waveguide. These are well-known results and we present it for the convenience of the reader. The dispersion equation is derived. We refer to the following works [1, 16, 19, 40, 64].

In Chapter 11 TE wave propagation in a circle cylindrical waveguide with Kerr nonlinearity is considered. For cylindrical waveguides we derive DEs as well, but we use integral equations technique instead of studying differential equations as in Chapters 3,4,6–8. The existence of eigenvalues is proved for sufficiently small the nonlinearity coefficient. Some results of the chapter were published in [49, 54–57].

In Chapter 12 we discuss TM wave propagation in a circle cylindrical waveguide. These are well-known results and we present it for the convenience of the reader. The dispersion equation is derived. It is also possible to see [1, 16, 19, 40, 64].

In Chapter 13 we consider the problem of TM wave propagation in a circle cylindrical waveguide with Kerr nonlinearity. In other respects the brief description of this chapter repeats the description of Chapter 11. Some results of the chapter were published in [58–60].

PART I

BOUNDARY EIGENVALUE PROBLEMS FOR THE MAXWELL EQUATIONS IN LAYERS

CHAPTER 1

TE AND TM WAVES GUIDED BY A LAYER

Let us consider electromagnetic waves propagating through a homogeneous isotropic nonmagnetic dielectric layer. The layer is located between two half-spaces: $x < 0$ and $x > h$ in Cartesian coordinate system $Oxyz$. The half-spaces are filled with isotropic nonmagnetic media without any sources and characterized by permittivities $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, respectively, where ε_0 is the permittivity of free space. Assume that everywhere $\mu = \mu_0$, where μ_0 is the permeability of free space.

The electromagnetic field depends on time harmonically [17]

$$\begin{aligned}\tilde{\mathbf{E}}(x, y, z, t) &= \mathbf{E}_+(x, y, z)\cos\omega t + \mathbf{E}_-(x, y, z)\sin\omega t,\\ \tilde{\mathbf{H}}(x, y, z, t) &= \mathbf{H}_+(x, y, z)\cos\omega t + \mathbf{H}_-(x, y, z)\sin\omega t,\end{aligned}$$

where ω is the circular frequency; $\tilde{\mathbf{E}}$, $\mathbf{E}_+$, $\mathbf{E}_-$, $\tilde{\mathbf{H}}$, $\mathbf{H}_+$, $\mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_+ + i\mathbf{E}_-,\\ \mathbf{H} &= \mathbf{H}_+ + i\mathbf{H}_-,\end{aligned}$$

where

$$\mathbf{E} = (E_x, E_y, E_z)^T, \quad \mathbf{H} = (H_x, H_y, H_z)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and

$$\begin{aligned}&E_x = E_x(x, y, z), \quad E_y = E_y(x, y, z), \quad E_z = E_z(x, y, z),\\ &H_x = H_x(x, y, z), \quad H_y = H_y(x, y, z), \quad H_z = H_z(x, y, z).\end{aligned}$$

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential field components on the media interfaces $x = 0$, $x = h$ and the radiation condition at infinity: the electromagnetic field exponentially decays as $|x| \to \infty$ in the domains $x < 0$ and $x > h$.

The permittivity in the layer is described by the diagonal tensor

$$\tilde{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix},$$

where ε_{xx}, ε_{yy}, and ε_{zz} are constants.

The solutions to the Maxwell equations are sought in the entire space.

The geometry of the problem is shown in Fig. 1. The layer is infinite along axes Oy and Oz.

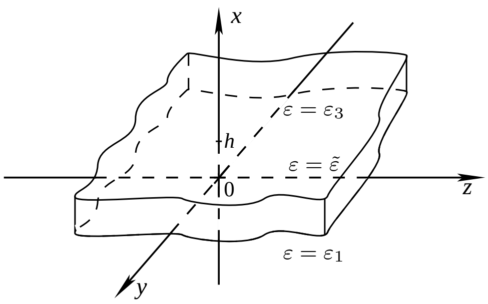

Fig. 1.

Rewrite system (1) in the coordinate form

$$\begin{cases} \dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z} = -i\omega\varepsilon_{xx}E_x, \\ \dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x} = -i\omega\varepsilon_{yy}E_y, \\ \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = -i\omega\varepsilon_{zz}E_z, \end{cases} \qquad \begin{cases} \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} = i\omega\mu H_x, \\ \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} = i\omega\mu H_y, \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = i\omega\mu H_z. \end{cases} \tag{2}$$

We seek surface waves propagating along z boundary of the layer. For such waves components of the fields have the form

$$\begin{array}{lll} E_x = E_x(x,y)e^{i\gamma z}, & E_y = E_y(x,y)e^{i\gamma z}, & E_z = E_z(x,y)e^{i\gamma z}, \\ H_x = H_x(x,y)e^{i\gamma z}, & H_y = H_y(x,y)e^{i\gamma z}, & H_z = H_z(x,y)e^{i\gamma z}, \end{array} \tag{3}$$

where γ is the propagation constant.

Substituting components (3) into system (1) we obtain

$$\begin{cases} \frac{\partial H_z}{\partial y} - i\gamma H_y = -i\omega\varepsilon_{xx}E_x, \\ i\gamma H_x - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_{yy}E_y, \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\omega\varepsilon_{zz}E_z, \end{cases} \qquad \begin{cases} \frac{\partial E_z}{\partial y} - i\gamma E_y = i\omega\mu H_x, \\ i\gamma E_x - \frac{\partial E_z}{\partial x} = i\omega\mu H_y, \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu H_z. \end{cases} \tag{4}$$

From the first and the fifth equations of this system we obtain

$$E_x = \frac{\omega\mu}{ik_{xx}^2}\left(\frac{\gamma}{\omega\mu}\frac{\partial E_z}{\partial x} + \frac{\partial H_z}{\partial y}\right), \quad H_y = \frac{\omega\varepsilon_{xx}}{ik_{xx}^2}\left(\frac{\partial E_z}{\partial x} + \frac{\gamma}{\omega\varepsilon_{xx}}\frac{\partial H_z}{\partial y}\right), \tag{5}$$

where $k_{xx}^2 = \gamma^2 - \omega^2\mu\varepsilon_{xx}$.

From the second and the forth equations of the latter system we obtain

$$E_y = -\frac{\omega\mu}{ik_{yy}^2}\left(\frac{\partial H_z}{\partial x} - \frac{\gamma}{\omega\mu}\frac{\partial E_z}{\partial y}\right), \quad H_x = \frac{\omega\varepsilon_{yy}}{ik_{xx}^2}\left(\frac{\gamma}{\omega\varepsilon_{yy}}\frac{\partial H_z}{\partial x} - \frac{\partial E_z}{\partial y}\right), \tag{6}$$

where $k_{yy}^2 = \gamma^2 - \omega^2\mu\varepsilon_{yy}$.

It is easy to see from formulas (5) and (6) that all the components are expressed through only two components E_z and H_z.

What is more, from formulas (5) and (6) we can see that the electromagnetic field can be represented as a linear combination of two fields (here the linearity of the Maxwell equations is essential)

$$\begin{aligned} \mathbf{E} &= (0, E_y, 0)^T + (E_x, 0, E_z)^T, \\ \mathbf{H} &= (H_x, 0, H_z)^T + (0, E_y, 0)^T. \end{aligned}$$

This means that in the case of the harmonic fields dependence on z we can consider two simpler electromagnetic fields

$$\mathbf{E} = (0, E_y, 0)^T, \quad \mathbf{H} = (H_x, 0, H_z)^T \tag{7}$$

and

$$\mathbf{E} = (E_x, 0, E_z)^T, \quad \mathbf{H} = (0, H_y, 0)^T \tag{8}$$

instead of one electromagnetic field

$$\mathbf{E} = (E_x, E_y, E_z)^T, \quad \mathbf{H} = (H_x, H_y, H_z)^T.$$

Electromagnetic waves (7) and (8) are called TE-polarized electromagnetic waves[1], or simply TE waves and TM-polarized electromagnetic waves[2], or simply TM waves, respectively.

Let us see what corollaries from Maxwell equations (1) and fields (7), (8) are implied.

Substituting fields (7) with components

$$E_y = E_y(x, y)e^{i\gamma z}, \;\; H_x = H_x(x, y)e^{i\gamma z}, \;\; H_z = H_z(x, y)e^{i\gamma z}$$

into system (1). We obtain

$$\begin{cases} \frac{\partial H_z}{\partial y} = 0, \\ i\gamma H_x - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_{yy}E_y, \\ -\frac{\partial H_x}{\partial y} = 0, \end{cases} \qquad \begin{cases} -i\gamma E_y = i\omega\mu H_x, \\ 0 = 0, \\ \frac{\partial E_y}{\partial x} = i\omega\mu H_z. \end{cases}$$

It follows from the first and the third equations of the system that H_z and H_x do not depend on y. This implies that E_y does not depend on z.

Thus we can conclude that in the case of the harmonic fields dependence on z the components of TE waves have the form

$$E_y = E_y(x)e^{i\gamma z}, \;\; H_x = H_x(x)e^{i\gamma z}, \;\; H_z = H_z(x)e^{i\gamma z}. \tag{9}$$

Substituting fields (8) with components

$$E_x = E_x(x, y)e^{i\gamma z}, \;\; E_z = E_z(x, y)e^{i\gamma z}, \;\; H_y = H_y(x, y)e^{i\gamma z}$$

into system (1). We obtain

$$\begin{cases} -i\gamma H_y = -i\omega\varepsilon_{xx}E_x, \\ 0 = 0, \\ \frac{\partial H_y}{\partial x} = -i\omega\varepsilon_{zz}E_z, \end{cases} \qquad \begin{cases} \frac{\partial E_z}{\partial y} = 0, \\ i\gamma E_x - \frac{\partial E_z}{\partial x} = i\omega\mu H_y, \\ -\frac{\partial E_x}{\partial y} = 0. \end{cases}$$

[1] transverse-electric.
[2] transverse-magnetic.

It follows from the forth and the sixth equations of the system that E_z and E_x do not depend on y. This implies that H_y does not depend on z.

Thus we can conclude that in the case of the harmonic fields dependence on z the components of TM waves have the form

$$E_x = E_x(x)e^{i\gamma z}, \quad E_z = E_z(x)e^{i\gamma z}, \quad H_y = H_y(x)e^{i\gamma z}. \qquad (10)$$

Representations (9) and (10) for TE and TM waves give us an opportunity to pass from partial differential equations (1) to ordinary differential equations.

As it is known, in homogeneous guided structures, like a layer with constant permittivity, any electromagnetic wave can be represented as a superposition of TE and TM waves [43]. This circumstance allows to study electromagnetic wave propagation in a linear layer only for polarized waves. This makes the analysis of the propagation quite simple. For a layer with nonlinear permittivity (when the permittivity depends on the electric field intensity) the general solution can not be represented as a superposition of TE and TM waves. So in the nonlinear case the problem does not break up into two simpler problems. Therefore studying the propagation of TE and TM waves in nonlinear layers we, generally speaking, find only specific solutions of Maxwell equations (1). These solutions correspond to TE and TM waves.

CHAPTER 2

TE WAVE PROPAGATION IN A LINEAR LAYER

§1. Statement of the Problem

Let us consider electromagnetic waves propagating through a homogeneous isotropic nonmagnetic dielectric layer. The layer is located between two half-spaces: $x < 0$ and $x > h$ in Cartesian coordinate system $Oxyz$. The half-spaces are filled with isotropic nonmagnetic media without any sources and characterized by permittivities $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, respectively, where ε_0 is the permittivity of free space[1]. Assume that everywhere $\mu = \mu_0$, where μ_0 is the permeability of free space.

The electromagnetic field depends on time harmonically [17]

$$\tilde{\mathbf{E}}(x, y, z, t) = \mathbf{E}_+(x, y, z)\cos\omega t + \mathbf{E}_-(x, y, z)\sin\omega t,$$
$$\tilde{\mathbf{H}}(x, y, z, t) = \mathbf{H}_+(x, y, z)\cos\omega t + \mathbf{H}_-(x, y, z)\sin\omega t,$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_+,\ \mathbf{E}_-,\ \tilde{\mathbf{H}},\ \mathbf{H}_+,\ \mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = (E_x, E_y, E_z)^T, \quad \mathbf{H} = (H_x, H_y, H_z)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

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[1] Generally, conditions $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$ are not necessary. They are not used for derivation of DEs, but they are useful for DEs' solvability analysis.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential field components on the media interfaces $x = 0$, $x = h$ and the radiation condition at infinity: the electromagnetic field exponentially decays as $|x| \to \infty$ in the domains $x < 0$ and $x > h$.

The permittivity inside the layer is $\varepsilon = \varepsilon_2$.

The solutions to the Maxwell equations are sought in the entire space.

The geometry of the problem is shown in Fig. 1.

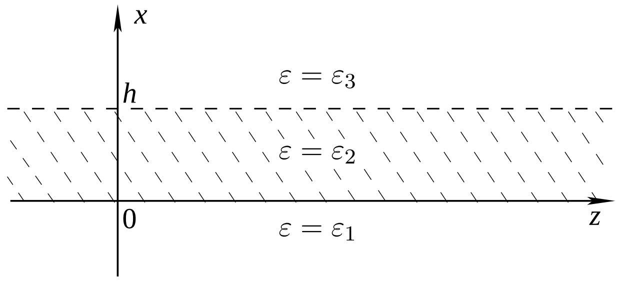


Fig. 1.

§2. TE Waves

Let us consider TE waves

$$\mathbf{E} = (0, E_y, 0)^T, \quad \mathbf{H} = (H_x, 0, H_z)^T,$$

where $E_y = E_y(x, y, z)$, $H_x = H_x(x, y, z)$, and $H_z = H_z(x, y, z)$.

Substituting the fields into Maxwell equations (1) we obtain

$$\begin{cases} \frac{\partial H_z}{\partial y} = 0, \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon E_y, \\ \frac{\partial H_x}{\partial y} = 0, \\ \frac{\partial E_y}{\partial z} = -i\omega\mu H_x, \\ \frac{\partial E_y}{\partial x} = i\omega\mu H_z. \end{cases}$$

It is obvious from the first and the third equations of this system that H_z and H_x do not depend on y. This implies that E_y does not depend on y.

Waves propagating along medium interface z depend on z harmonically. This means that the fields components have the form

$$E_y = E_y(x)e^{i\gamma z},\ H_x = H_x(x)e^{i\gamma z},\ H_z = H_z(x)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system

$$\begin{cases} i\gamma H_x(x) - H_z'(x) = -i\omega\varepsilon E_y(x), \\ -i\gamma E_y(x) = i\omega\mu H_x(x), \\ E_y'(x) = i\omega\mu H_z(x), \end{cases} \tag{2}$$

where $(\,\cdot\,)' \equiv \frac{d}{dx}$.

After simple transformation of system (2) we obtain

$$\gamma^2 E_y(x) - E_y''(x) = \omega^2\mu\varepsilon E_y(x).$$

Let us denote by $k_0^2 := \omega^2\mu\varepsilon_0$, and perform the normalization according to the formulas $\tilde{x} = k_0 x$, $\frac{d}{dx} = k_0\frac{d}{d\tilde{x}}$, $\tilde{\gamma} = \frac{\gamma}{k_0}$, $\tilde{\varepsilon}_j = \frac{\varepsilon_j}{\varepsilon_0}$ ($j = 1, 2, 3$). Denoting by $Y(\tilde{x}) := E_y(\tilde{x})$ and omitting the tilde symbol from system (2) we have

$$Y''(x) = \gamma^2 Y(x) - \varepsilon Y(x). \tag{3}$$

Introducing the function $Z(x) := Y'(x)$ we can consider (3) as the following system of equations

$$\begin{cases} Y'(x) = Z(x), \\ Z'(x) = \left(\gamma^2 - \varepsilon\right) Y(x). \end{cases} \tag{4}$$

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along boundaries of the layer $0 < x < h$, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of spectral parameter γ such that real solutions $Y(x)$ and $Z(x)$ to system (4) exist.

Note. We consider that γ is a real value, but in the linear case it is possible to consider the spectral parameter γ as a complex value. In nonlinear cases under our approach it is impossible to use complex values of γ.

Also we assume that functions Y and Z are sufficiently smooth

$$\begin{aligned} &Y(x) \in C(-\infty,+\infty)\cap \\ &\qquad \cap\, C^1(-\infty,+\infty)\cap C^2(-\infty,0)\cap C^2(0,h)\cap C^2(h,+\infty), \\ &Z(x) \in C(-\infty,+\infty)\cap C^1(-\infty,0)\cap C^1(0,h)\cap C^1(h,+\infty). \end{aligned}$$

Physical nature of the problem implies these conditions.

We will seek γ under condition $\max(\varepsilon_1,\varepsilon_3) < \gamma^2 < \varepsilon_2$. It should be noticed that condition $\gamma^2 > \max(\varepsilon_1,\varepsilon_3)$ holds if at least one of the values ε_1 or ε_3 more than zero. If both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem

In the domain $x < 0$ we have $\varepsilon = \varepsilon_1$. From system (4) we obtain $Y'' = \left(\gamma^2 - \varepsilon_1\right) Y$. Its general solution is

$$Y(x) = A_1 e^{-\sqrt{\gamma^2-\varepsilon_1}x} + A e^{\sqrt{\gamma^2-\varepsilon_1}x}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} &Y(x) = A e^{x\sqrt{\gamma^2-\varepsilon_1}}, \\ &Z\left(x\right) = A\sqrt{\gamma^2-\varepsilon_1}\, e^{x\sqrt{\gamma^2-\varepsilon_1}}. \end{aligned} \tag{5}$$

We assume that $\gamma^2 - \varepsilon_1 > 0$, otherwise it will be impossible to satisfy the radiation condition.

In the domain $x > h$ we have $\varepsilon = \varepsilon_3$. From system (4) we obtain $Y'' = \left(\gamma^2 - \varepsilon_3\right) Y$. Its general solution is

$$Y(x) = B_1 e^{(x-h)\sqrt{\gamma^2-\varepsilon_3}} + B e^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} Y(x) &= Be^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}}, \\ Z(x) &= -\sqrt{\gamma^2-\varepsilon_3}Be^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}}. \end{aligned} \tag{6}$$

Here for the same reason as above we consider that $\gamma^2 - \varepsilon_3 > 0$.

Constants A and B in (5) and (6) are defined by transmission conditions and initial conditions.

Inside the layer $0 < x < h$ we have $\varepsilon = \varepsilon_3$. From system (4) we obtain $Y'' = (\gamma^2 - \varepsilon_2)Y$. It is possible to consider two cases:

a) $\gamma^2 - \varepsilon_2 > 0$; and the general solution of system (4) inside the layer is

$$\begin{aligned} Y(x) &= C_1e^{-x\sqrt{\gamma^2-\varepsilon_2}} + C_2e^{x\sqrt{\gamma^2-\varepsilon_2}}, \\ Z(x) &= \sqrt{\gamma^2-\varepsilon_2}\left(-C_1e^{-x\sqrt{\gamma^2-\varepsilon_2}} + C_2e^{x\sqrt{\gamma^2-\varepsilon_2}}\right); \end{aligned} \tag{7}$$

b) $\gamma^2 - \varepsilon_2 < 0$; and the general solution of system (4) inside the layer is

$$\begin{aligned} Y(x) &= C_1 \sin x\sqrt{\varepsilon_2-\gamma^2} + C_2 \cos x\sqrt{\varepsilon_2-\gamma^2}, \\ Z(x) &= \sqrt{\varepsilon_2-\gamma^2}\left(C_1 \cos x\sqrt{\varepsilon_2-\gamma^2} - C_2 \sin x\sqrt{\varepsilon_2-\gamma^2}\right). \end{aligned} \tag{8}$$

§4. Transmission Conditions

Tangential components of an electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are E_y and H_z. Hence we obtain

$$\begin{aligned} E_y(h+0) &= E_y(h-0), \quad E_y(0-0) = E_y(0+0), \\ H_z(h+0) &= H_z(h-0), \quad H_z(0-0) = H_z(0+0). \end{aligned}$$

The continuity conditions for components E_y, H_z and formulas (2), (4) imply the transmission conditions for Y, Z

$$[Y]_{x=0} = 0, \quad [Y]_{x=h} = 0, \quad [Z]_{x=0} = 0, \quad [Z]_{x=h} = 0, \tag{9}$$

where $[f]_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$ denotes a jump of the function f at the interface.

Denote by $Y_0 := Y(0-0)$, $Y_h := Y(h+0)$, $Z_0 := Z(0-0)$, and $Z_h := Z(h+0)$. Then, we obtain $A = Y_0$, $B = Y_h$ and

$$Z_h = -\sqrt{\gamma^2 - \varepsilon_3} Y_h \quad Z_0 = \sqrt{\gamma^2 - \varepsilon_1} Y_0.$$

The constant Y_h is supposed to be known (initial condition).

In case (a) from transmission conditions (9) and solutions (5)–(7) we obtain the system

$$\begin{cases} A = C_1 + C_2, \\ B = C_1 e^{-h\sqrt{\gamma^2-\varepsilon_2}} + C_2 e^{h\sqrt{\gamma^2-\varepsilon_2}}, \\ \sqrt{\gamma^2 - \varepsilon_1} A = \sqrt{\gamma^2 - \varepsilon_2} \left(-C_1 + C_2\right), \\ -\sqrt{\gamma^2 - \varepsilon_3} B = \sqrt{\gamma^2 - \varepsilon_2} \left(-C_1 e^{-h\sqrt{\gamma^2-\varepsilon_2}} + C_2 e^{h\sqrt{\gamma^2-\varepsilon_2}}\right). \end{cases}$$

Solving this system we obtain the DE

$$\frac{\sqrt{\gamma^2 - \varepsilon_2} - \sqrt{\gamma^2 - \varepsilon_1}}{\sqrt{\gamma^2 - \varepsilon_2} + \sqrt{\gamma^2 - \varepsilon_1}} \cdot \frac{\sqrt{\gamma^2 - \varepsilon_2} - \sqrt{\gamma^2 - \varepsilon_3}}{\sqrt{\gamma^2 - \varepsilon_2} + \sqrt{\gamma^2 - \varepsilon_3}} = e^{2h\sqrt{\gamma^2-\varepsilon_2}}, \quad (10)$$

where $\gamma^2 - \varepsilon_1 > 0$, $\gamma^2 - \varepsilon_2 > 0$, $\gamma^2 - \varepsilon_3 > 0$.

In case (b) from transmission conditions (9) and solutions (5), (6), (8) we obtain the system

$$\begin{cases} A = C_2, \\ B = C_1 \sin\sqrt{\varepsilon_2 - \gamma^2} h + C_2 \cos\sqrt{\varepsilon_2 - \gamma^2} h, \\ \sqrt{\gamma^2 - \varepsilon_1} A = C_1 \sqrt{\varepsilon_2 - \gamma^2}, \\ -\sqrt{\gamma^2 - \varepsilon_3} B = \\ \qquad = \sqrt{\gamma^2 - \varepsilon_2} \left(-C_1 \cos\sqrt{\varepsilon_2 - \gamma^2} h - C_2 \sin\sqrt{\varepsilon_2 - \gamma^2} h\right). \end{cases}$$

From this system we find

$$\frac{1}{\sqrt{\varepsilon_2 - \gamma^2}} \frac{\varepsilon_2 - \gamma^2 - \sqrt{\gamma^2 - \varepsilon_1}\sqrt{\gamma^2 - \varepsilon_3}}{\sqrt{\gamma^2 - \varepsilon_1} + \sqrt{\gamma^2 - \varepsilon_3}} \sin\sqrt{\varepsilon_2 - \gamma^2} h = \\ = \cos\sqrt{\varepsilon_2 - \gamma^2} h. \quad (11)$$

If $\cos\sqrt{\varepsilon_2-\gamma^2}h \neq 0$, then we obtain the well-known DE

$$\operatorname{tg}\left(\sqrt{\varepsilon_2-\gamma^2}h\right) = \frac{\sqrt{\varepsilon_2-\gamma^2}\left(\sqrt{\gamma^2-\varepsilon_1}+\sqrt{\gamma^2-\varepsilon_3}\right)}{\varepsilon_2-\gamma^2-\sqrt{\gamma^2-\varepsilon_1}\sqrt{\gamma^2-\varepsilon_3}}, \qquad (12)$$

where $\gamma^2-\varepsilon_1>0$, $\varepsilon_2-\gamma^2>0$, $\gamma^2-\varepsilon_3>0$.

If $\cos\sqrt{\varepsilon_2-\gamma^2}h = 0$, then we can find eigenvalues in explicit form.

Let $\cos\sqrt{\varepsilon_2-\gamma^2}h = 0$, then

$$\sqrt{\varepsilon_2-\gamma^2} = \frac{\pi\,(2n+1)}{2h} \quad \text{and} \quad \gamma^2 = \frac{4\varepsilon_2 h^2-\pi^2\,(2n+1)^2}{4h^2}.$$

From equation (11) we obtain $\varepsilon_2-\gamma^2 = \sqrt{\gamma^2-\varepsilon_1}\sqrt{\gamma^2-\varepsilon_3}$. From this equation we find γ^2. After simple transformation we obtain

$$h = \frac{\pi\,(2n+1)}{2}\sqrt{\frac{2\varepsilon_2-\varepsilon_1-\varepsilon_3}{(\varepsilon_2-\varepsilon_1)\,(\varepsilon_2-\varepsilon_3)}}, \quad \gamma^2 = \frac{\varepsilon_2^2-\varepsilon_1\varepsilon_3}{2\varepsilon_2-\varepsilon_1-\varepsilon_3}, \quad (13)$$

and under conditions $\gamma^2-\varepsilon_1>0$, $\varepsilon_2-\gamma^2>0$, $\gamma^2-\varepsilon_3>0$ the radicand in (13) is nonnegative.

In the simplest case when $\varepsilon_1=\varepsilon_3=\varepsilon$ from (13) we obtain

$$h = \frac{(2n+1)\pi}{\sqrt{2(\varepsilon_2-\varepsilon)}}, \quad \gamma^2 = \frac{\varepsilon_2+\varepsilon}{2}.$$

Equation (12) can be formally derived from (10). Indeed, if we simply change in (10) $\gamma^2-\varepsilon_2$ by $-(\varepsilon_2-\gamma^2)$, take into account occurred imaginary unit, then we obtain (12). In the same way it is possible to derive (10) from (12).

§5. Analysis of Dispersion Equations

It follows from conditions $\gamma^2-\varepsilon_1>0$ and $\gamma^2-\varepsilon_3>0$ that ε_1 and ε_3 can be values of arbitrary sign (this is mentioned in §1) in both DEs (10), (12). Let us consider in detail the case when $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, where ε_0 is the permittivity of free space. It is clear that for classical DE (12) we can not study metamaterial in the layer due to the DE is derived under condition $\varepsilon_2 - \gamma^2 > 0$. This means that $\varepsilon_2 > 0$. Studying this DE for metamaterials is of no interest.

Conditions $\gamma^2 - \varepsilon_1 > 0$ and $\gamma^2 - \varepsilon_3 > 0$ imply immediately that

$$\gamma^2 > \max(\varepsilon_1, \varepsilon_3),$$

if at least one of the values ε_1 or ε_3 is more than zero. If both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$, then

$$\gamma^2 > 0.$$

Let us pass to the analysis of DEs (10) and (12).

Rewrite equation (10) in the following form

$$h = \frac{\ln\left(\frac{\sqrt{\gamma^2-\varepsilon_2}-\sqrt{\gamma^2-\varepsilon_1}}{\sqrt{\gamma^2-\varepsilon_2}+\sqrt{\gamma^2-\varepsilon_1}} \cdot \frac{\sqrt{\gamma^2-\varepsilon_2}-\sqrt{\gamma^2-\varepsilon_3}}{\sqrt{\gamma^2-\varepsilon_2}+\sqrt{\gamma^2-\varepsilon_3}}\right)}{2\sqrt{\gamma^2-\varepsilon_2}} + \frac{i\pi k}{\sqrt{\gamma^2-\varepsilon_2}}, \tag{14}$$

where $k \in \mathbb{Z}$.

It is easy to see in (14) that

$$\left|\frac{\sqrt{\gamma^2-\varepsilon_2}-\sqrt{\gamma^2-\varepsilon_1}}{\sqrt{\gamma^2-\varepsilon_2}+\sqrt{\gamma^2-\varepsilon_1}}\right| < 1, \quad \left|\frac{\sqrt{\gamma^2-\varepsilon_2}-\sqrt{\gamma^2-\varepsilon_3}}{\sqrt{\gamma^2-\varepsilon_2}+\sqrt{\gamma^2-\varepsilon_3}}\right| < 1.$$

Under this condition and $\gamma^2 - \varepsilon_2 > 0$ equation (14) implies that $h < 0$. But such a case is impossible due to the value h denotes thickness of the layer.

Now let us consider equation (12). This equation is well-known (see, for example [64]). The condition $\varepsilon_2 - \gamma^2 > 0$ immediately implies that $\varepsilon_2 > 0$. From this and from analysis of DE (10) it follows that there are no waves in the layer with negative permittivity in the case of TE waves!

It is easy to show that equation (12) can be rewritten in the following form

$$h = \frac{1}{\sqrt{\varepsilon_2-\gamma^2}}\left(\operatorname{arctg}\frac{\sqrt{\varepsilon_2-\gamma^2}\left(\sqrt{\gamma^2-\varepsilon_1}+\sqrt{\gamma^2-\varepsilon_3}\right)}{\varepsilon_2-\gamma^2-\sqrt{\gamma^2-\varepsilon_1}\sqrt{\gamma^2-\varepsilon_3}} + \pi(n+1)\right), \tag{15}$$

where $n \geq -1$ is an integer.

Indeed, as $|\operatorname{arctg} x| < \frac{\pi}{2}$ then $h < 0$ when $n + 1 \leq -1$. This implies that $n + 1 \geq 0$, and we have $n \geq -1$[1].

It is obvious from formula (15) that the line $\gamma^2 = \varepsilon_2$ is an asymptote: $h^* = \lim\limits_{\gamma^2 \to \varepsilon_2 - 0} h(\gamma) = +\infty$. If $\gamma^2 > \varepsilon_2$, then we obtain imaginary value for h. From conditions $\gamma^2 > \varepsilon_2$ and $\gamma^2 > \max(\varepsilon_1, \varepsilon_3)$ the important conclusion follows: in the case of TE waves in a linear layer the spectral parameter γ satisfies the following inequalities

$$\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2,$$

where at least one of the values ε_1 or ε_3 is positive. If both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$, then the spectral parameter γ satisfies the following inequalities

$$0 < \gamma^2 < \varepsilon_2.$$

Introduce the notation $\varepsilon^* := \max(\varepsilon_1, \varepsilon_3)$, $\varepsilon_* := \min(\varepsilon_1, \varepsilon_3)$, and $h_* := \lim\limits_{\gamma^2 \to \varepsilon^*} h(\gamma)$. Then $h_* = \frac{1}{\sqrt{\varepsilon_2 - \varepsilon^*}} \operatorname{arctg} \sqrt{\frac{\varepsilon^* - \varepsilon_*}{\varepsilon_2 - \varepsilon^*}}$.

It is obvious that $0 \leq h_* < +\infty$. The value h_* increases if the value $\varepsilon_2 - \varepsilon^*$ decreases.

For DE (15) we obtain the following

Conclusion. There are only finite number of waves in a layer with constant permittivity. This finite number is equal to the number of eigenvalues (solutions of the DE). The quantity of waves increases if the value h increases. If $\varepsilon_* \neq \varepsilon^*$ (in other words, if $\varepsilon_1 \neq \varepsilon_3$), then

---

[1] The behavior of the function $\operatorname{arctg} \frac{\sqrt{\varepsilon_2 - \gamma^2}\left(\sqrt{\gamma^2 - \varepsilon_1} + \sqrt{\gamma^2 - \varepsilon_3}\right)}{\varepsilon_2 - \gamma^2 - \sqrt{\gamma^2 - \varepsilon_1}\sqrt{\gamma^2 - \varepsilon_3}}$ should be noticed. The denominator $\varepsilon_2 - \gamma^2 - \sqrt{\gamma^2 - \varepsilon_1}\sqrt{\gamma^2 - \varepsilon_3}$ vanishes in the certain point $\gamma_*^2 \in (\max(\varepsilon_1, \varepsilon_3), \varepsilon_2)$. This means that the function arctg discontinues at this point. It is well-known that this discontinuity of the first kind and the jump is equal to π. This implies that each dispersion curve (DC) consists of two pieces: the first piece corresponds to $\gamma^2 \in (\max(\varepsilon_1, \varepsilon_3), \gamma_*^2)$, and the second one corresponds to $\gamma^2 \in (\gamma_*^2, \varepsilon_2)$. If we consider a particular DC, then the first piece is a part of this DC and the second one is a part of the next DC. Thus the whole DC consists of the first piece, which belongs to this very DC and of the second piece, which belongs to the previous DC. These two pieces taken together form the continuous DC. If $n = -1$ in (15), then DC $h \equiv h(\gamma)$ defining by (15), partly lies in half-plane $h < 0$ and partly lies in half-plane $h > 0$. We consider only the part of the DC that lies in the half-plane $h > 0$.

there is $h_* > 0$ such that there are no waves in the layer with $h < h_*$. This conclusion holds only for a linear waveguide structure.

The DCs are shown in Fig. 2,3.

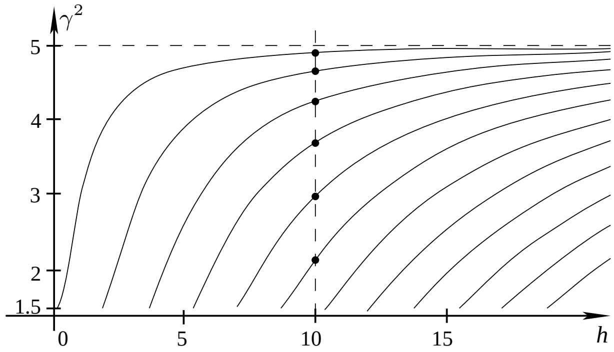


Fig. 2. $\varepsilon_1 = 1$, $\varepsilon_2 = 5$, $\varepsilon_3 = 1.5$

The quantity of eigenvalues is defined in the following way: for example, in Fig. 2 we draw the line which corresponds to layer's thickness h (the dashed line $h = 10$). The quantity of eigenvalues is equal to the quantity of points where the line intersects the DCs (there are six intersections in Fig. 2, they are marked by black dots). It is easy to understand from Fig. 2 why the quantity of propagating waves (quantity of eigenvalues) increases if the layer's thickness h increases.

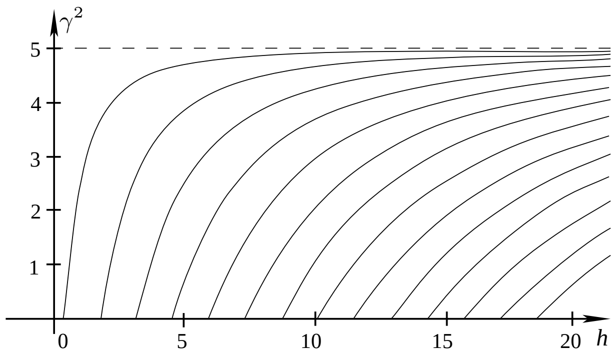


Fig. 3. $\varepsilon_1 = -1$, $\varepsilon_2 = 5$, $\varepsilon_3 = -1.5$

In the case $\varepsilon_1 < 0$, $\varepsilon_3 > 0$ or $\varepsilon_1 > 0$, $\varepsilon_3 < 0$, the behavior of the DCs is the same as it is shown in Fig. 3.

CHAPTER 3

TE WAVE PROPAGATION IN A LAYER WITH ARBITRARY NONLINEARITY

§1. Statement of the Problem

Let us consider electromagnetic waves propagating through a homogeneous isotropic nonmagnetic dielectric layer. The layer is located between two half-spaces: $x < 0$ and $x > h$ in Cartesian coordinate system $Oxyz$. The half-spaces are filled with isotropic nonmagnetic media without any sources and characterized by permittivities $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, respectively, where ε_0 is the permittivity of free space[1]. Assume that everywhere $\mu = \mu_0$, where μ_0 is the permeability of free space.

The electromagnetic field depends on time harmonically [17]

$$\tilde{\mathbf{E}}\,(x, y, z, t) = \mathbf{E}_+\,(x, y, z) \cos \omega t + \mathbf{E}_-\,(x, y, z) \sin \omega t,$$
$$\tilde{\mathbf{H}}\,(x, y, z, t) = \mathbf{H}_+\,(x, y, z) \cos \omega t + \mathbf{H}_-\,(x, y, z) \sin \omega t,$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_+,\ \mathbf{E}_-,\ \tilde{\mathbf{H}},\ \mathbf{H}_+,\ \mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = (E_x, E_y, E_z)^T\,, \quad \mathbf{H} = (H_x, H_y, H_z)^T\,,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

[1]Generally, conditions $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$ are not necessary. They are not used for derivation of DEs, but they are useful for DEs' solvability analysis.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential field components on the media interfaces $x = 0$, $x = h$ and the radiation condition at infinity: the electromagnetic field exponentially decays as $|x| \to \infty$ in the domains $x < 0$ and $x > h$.

The permittivity inside the layer has the form

$$\varepsilon = \varepsilon_2 + \varepsilon_0 f\left(|\mathbf{E}|^2\right),$$

where f is an analytical function[1].

Also we assume that $\varepsilon_2 > \max(\varepsilon_1, \varepsilon_3)$.

The solution to the Maxwell equations are sought in the entire space.

The geometry of the problem is shown in Fig. 1.

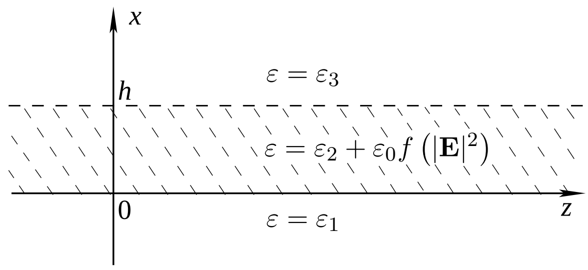


Fig. 1.

§2. TE Waves

Let us consider TE waves

$$\mathbf{E} = (0, E_y, 0)^T, \quad \mathbf{H} = (H_x, 0, H_z)^T,$$

where $E_y = E_y(x, y, z)$, $H_x = H_x(x, y, z)$, and $H_z = H_z(x, y, z)$.

[1] Everywhere below when we consider an analytical function we mean that it is the analytical function of a real variable (see the end of § 2).

Substituting the fields into Maxwell equations (1) we obtain

$$\begin{cases} \frac{\partial H_z}{\partial y} = 0, \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon E_y, \\ \frac{\partial H_x}{\partial y} = 0, \\ \frac{\partial E_y}{\partial z} = -i\omega\mu H_x, \\ \frac{\partial E_y}{\partial x} = i\omega\mu H_z. \end{cases}$$

It is obvious from the first and the third equations of this system that H_z and H_x do not depend on y. This implies that E_y does not depend on y.

Waves propagating along medium interface z depend on z harmonically. This means that the fields components have the form

$$E_y = E_y(x)e^{i\gamma z},\ H_x = H_x(x)e^{i\gamma z},\ H_z = H_z(x)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system

$$\begin{cases} i\gamma H_x(x) - H_z'(x) = -i\omega\varepsilon E_y(x), \\ -i\gamma E_y(x) = i\omega\mu H_x(x), \\ E_y'(x) = i\omega\mu H_z(x), \end{cases} \tag{2}$$

where $(\ \cdot\ )' \equiv \frac{d}{dx}$.

After simple transformation of system (2) we obtain

$$\gamma^2 E_y(x) - E_y''(x) = \omega^2\mu\varepsilon E_y(x).$$

Let us denote by $k_0^2 := \omega^2\mu_0\varepsilon_0$, and perform the normalization according to the formulas $\tilde{x} = k_0 x$, $\frac{d}{dx} = k_0\frac{d}{d\tilde{x}}$, $\tilde{\gamma} = \frac{\gamma}{k_0}$, $\tilde{\varepsilon}_j = \frac{\varepsilon_j}{\varepsilon_0}$ ($j = 1, 2, 3$). Denoting by $Y(\tilde{x}) := E_y(\tilde{x})$ and omitting the tilde symbol, we have

$$Y''(x) = \gamma^2 Y(x) - \varepsilon Y(x). \tag{3}$$

Introducing the function $Z(x) := Y'(x)$ we can consider (3) as the following system of equations

$$\begin{cases} Y'(x) = Z(x), \\ Z'(x) = \left(\gamma^2 - \varepsilon\right) Y(x). \end{cases} \tag{4}$$

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along boundaries of the layer $0 < x < h$, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of spectral parameter γ such that real solutions $Y(x)$ and $Z(x)$ to system (4) exist[1] (see also the note on p. 23). We assume that

$$\varepsilon = \begin{cases} \varepsilon_1, & x < 0; \\ \varepsilon_2 + f\left(Y^2\right), & 0 < x < h; \\ \varepsilon_3, & x > h. \end{cases} \tag{5}$$

Also we assume that functions Y and Z are sufficiently smooth

$$\begin{aligned} &Y(x) \in C(-\infty, +\infty) \cap \\ &\qquad \cap C^1(-\infty, +\infty) \cap C^2(-\infty, 0) \cap C^2(0, h) \cap C^2(h, +\infty), \\ &Z(x) \in C(-\infty, +\infty) \cap C^1(-\infty, 0) \cap C^1(0, h) \cap C^1(h, +\infty). \end{aligned}$$

Physical nature of the problem implies these conditions.

It is clear that system (4) is an autonomous one with analytical right-hand sides with respect to Y and Z. It is well known (see [5]) that solutions Y, Z of such a system are also analytical functions with respect to independent variable. This is an important fact for the DEs' derivation.

We will seek γ under condition $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$.

[1] In this case $|\mathbf{E}|^2$ does not depend on z. Since $\mathbf{E} = \left(0, E_y(x)e^{i\gamma z}, 0\right) = e^{i\gamma z}\,(0, E_y, 0)$; therefore, $|\mathbf{E}| = \left|e^{i\gamma z}\right| \cdot |E_y|$. It is known that $\left|e^{i\gamma z}\right| = 1$ as $\operatorname{Im}\gamma = 0$. Let $\gamma = \gamma' + i\gamma''$. Then, we obtain $\left|e^{i\gamma z}\right| = \left|e^{i\gamma' z}\right| \cdot \left|e^{-\gamma'' z}\right| = e^{-\gamma'' z}$. If $\gamma'' \neq 0$, then $e^{-\gamma'' z}$ is a function on z. In this case the component E_y depends on z, but it contradicts to the choice of $E_y(x)$. So we can consider only real values of γ.

This condition corresponds to the classical problem of TE wave propagation in a linear layer, when $\varepsilon_1 \geq \varepsilon_0$, $\varepsilon_3 \geq \varepsilon_0$, and ε in the layer is equal to ε_2 and $\varepsilon_2 > \max(\varepsilon_1, \varepsilon_3)$. This condition naturally appears in that problem (see. Ch. 2), therefore we use it to derive the DEs for a nonlinear layer. In §6 the DE is obtained under the most general conditions. We also notice that condition $\gamma^2 > \max(\varepsilon_1, \varepsilon_3)$ holds if at least one of the values ε_1 or ε_3 more than zero. If both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem

In the domain $x < 0$ we have $\varepsilon = \varepsilon_1$. From system (4) we obtain $Y'' = \left(\gamma^2 - \varepsilon_1\right) Y$. Its general solution is

$$Y(x) = A_1 e^{-x\sqrt{\gamma^2-\varepsilon_1}} + A e^{x\sqrt{\gamma^2-\varepsilon_1}}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} Y(x) &= A \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right), \\ Z(x) &= A\sqrt{\gamma^2 - \varepsilon_1} \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right). \end{aligned} \tag{6}$$

We assume that $\gamma^2 - \varepsilon_1 > 0$ otherwise it will be impossible to satisfy the radiation condition.

In the domain $x > h$ we have $\varepsilon = \varepsilon_3$. From system (4) we obtain $Y'' = \left(\gamma^2 - \varepsilon_3\right) Y$. Its general solution is

$$Y(x) = B_1 e^{(x-h)\sqrt{\gamma^2-\varepsilon_3}} + B e^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} Y(x) &= B \exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right), \\ Z(x) &= -\sqrt{\gamma^2 - \varepsilon_3} B \exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right). \end{aligned} \tag{7}$$

Here for the same reason as above we consider that $\gamma^2 - \varepsilon_3 > 0$.

Constants A and B in (6) and (7) are defined by transmission conditions and initial conditions.

Inside the layer $0 < x < h$ system (4) has the form

$$\begin{cases} Y'(x) = Z(x), \\ Z'(x) = \left(\gamma^2 - \varepsilon_2 - f\left(Y^2\right)\right) Y(x). \end{cases} \tag{8}$$

System (8) has the first integral. So we can study the first-order equation (either the first or the second in (8)) with the first integral instead of second-order equation (3). Divide the second equation in (8) by the other one we obtain

$$ZdZ + \left(\varepsilon_2 - \gamma^2 + f\left(Y^2\right)\right) YdY = 0.$$

This equation is the total differential equation. Its general solution has the form

$$Z^2 + (\varepsilon_2 - \gamma^2)Y^2 + \varphi(Y^2) \equiv C, \tag{9}$$

where $\varphi(Y^2) = \int f(u)du\big|_{u=Y^2}$, and C is a constant of integration.

§4. Transmission Conditions and the Transmission Problem

Tangential components of an electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are E_y and H_z. Hence we obtain

$$E_y(h+0) = E_y(h-0), \quad E_y(0-0) = E_y(0+0),$$
$$H_z(h+0) = H_z(h-0), \quad H_z(0-0) = H_z(0+0).$$

The continuity conditions for components E_y, H_z and formulas (2), (4) imply the transmission conditions for Y, Z

$$[Y]_{x=0} = 0, \quad [Y]_{x=h} = 0, \quad [Z]_{x=0} = 0, \quad [Z]_{x=h} = 0, \tag{10}$$

where $[f]_{x=x_0} = \lim\limits_{x \to x_0 - 0} f(x) - \lim\limits_{x \to x_0 + 0} f(x)$ denotes a jump of the function f at the interface.

Denote by $Y_0 := Y(0-0)$, $Y_h := Y(h+0)$, $Z_0 := Z(0-0)$, and $Z_h := Z(h+0)$. Then, we obtain $A = Y_0$, $B = Y_h$ and

$$Z_0 = \sqrt{\gamma^2 - \varepsilon_1} Y_0, \quad Z_h = -\sqrt{\gamma^2 - \varepsilon_3} Y_h.$$

The constant Y_h is supposed to be known (initial condition).

We also suppose that functions $Y(x)$, $Z(x)$ satisfy the condition

$$Y(x) = O\left(\frac{1}{|x|}\right), \quad Z(x) = O\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty. \tag{11}$$

Let

$$\mathrm{D} = \begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & \frac{d}{dx} \end{pmatrix}, \mathbf{F}(X, Z) = \begin{pmatrix} X \\ Z \end{pmatrix}, \mathbf{G}(\mathbf{F}, \gamma) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

where $Y \equiv Y(x)$, $Z \equiv Z(x)$ are unknown functions, $G_1 \equiv G_1(\mathbf{F}, \gamma)$, $G_2 \equiv G_2(\mathbf{F}, \gamma)$ are right-hand sides of system (8). The value γ is a spectral parameter. Rewrite the problem using new notation.

For the half-space $x < 0$ and $\varepsilon = \varepsilon_1$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & 1 \\ \gamma^2 - \varepsilon_1 & 0 \end{pmatrix} \mathbf{F} = 0. \tag{12}$$

Inside the layer $0 < x < h$ and $\varepsilon = \varepsilon_2 + f\left(Y^2\right)$ we have

$$\mathrm{L}(\mathbf{F}, \gamma) \equiv \mathrm{D}\mathbf{F} - \mathbf{G}(\mathbf{F}, \gamma) = 0. \tag{13}$$

For the half-space $x > h$ and $\varepsilon = \varepsilon_3$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & 1 \\ \gamma^2 - \varepsilon_3 & 0 \end{pmatrix} \mathbf{F} = 0. \tag{14}$$

Using $Z_h = -\sqrt{\gamma^2 - \varepsilon_3} Y_h$ and first integral (9) we find the integration constant value $C_h^Y := C|_{x=h}$

$$C_h^Y = (\varepsilon_2 - \varepsilon_3) Y_h^2 + \varphi(Y_h^2).$$

Transmission conditions (10) and first integral (9) imply the equation with respect to Y_0

$$(\varepsilon_2 - \varepsilon_3) Y_h^2 + \varphi\left(Y_h^2\right) = (\varepsilon_2 - \varepsilon_1) Y_0^2 + \varphi\left(Y_0^2\right). \tag{15}$$

Let us formulate the transmission problem (it is possible to reformulate it as the boundary eigenvalue problem). *It is necessary to find eigenvalues* γ *and corresponding to them nonzero vectors* $\mathbf{F}$ *such that* $\mathbf{F}$ *satisfies to equations* (12)–(14). *Components* Y, Z *of vector* $\mathbf{F}$ *satisfy transmission conditions* (10), *condition* (11), *and* $Y(0) \equiv Y_0$ *is defined from equation* (15).

Definition 1. The value $\gamma = \gamma_0$ such that nonzero solution $\mathbf{F}$ to problem (12)–(14) exists under conditions (10), (11), and (15) is called an eigenvalue of the problem. Solution $\mathbf{F}$, corresponding to the eigenvalue is called an eigenvector of the problem, and components $Y(x)$ and $Z(x)$ of vector $\mathbf{F}$ are called eigenfunctions.

Note. Definition 1 is a nonclassical analog of the known definition of the characteristic number of a linear operator function nonlinearly depending on the spectral parameter [20]. This definition, on the one hand, is an extension of the classic definition of an eigenvalue to the case of a nonlinear operator function. On the other hand, it corresponds to the physical nature of the problem.

§5. Dispersion Equation

Introduce the new variables[1]

$$\tau(x) = \varepsilon_2 + Y^2(x), \quad \eta(x) = \frac{Z(x)}{Y(x)}. \tag{16}$$

From (16) we obtain

$$Y^2 = \tau - \varepsilon_2, \quad YZ = (\tau - \varepsilon_2)\eta, \quad Z^2 = (\tau - \varepsilon_2)\eta^2. \tag{17}$$

System (8) takes the form

$$\begin{cases} \tau' = 2(\tau - \varepsilon_2)\eta, \\ \eta' = \left(\gamma^2 - \varepsilon_2 - f(\tau - \varepsilon_2) - \eta^2\right). \end{cases} \tag{18}$$

First integral (9) has the form

$$(\tau - \varepsilon_2)\eta^2 + (\varepsilon_2 - \gamma^2)(\tau - \varepsilon_2) + \varphi(\tau - \varepsilon_2) \equiv C. \tag{19}$$

[1] Generally, if the nonlinearity function is a specific one, then it is possible to choose new variables in another way (see Ch. 7 and 8).

Generally speaking, equation (19) is a transcendental one with respect to τ. Its solution $\tau = \tau(\eta)$ can be easily expressed in explicit form only in exceptional cases.

If nonlinearity function f is a polynomial, then equation (19) is an algebraic equation with respect to τ. The polarization vector in constitutive relations in the Maxwell equations can be expanded into a series in $|\mathbf{E}|$. When we consider that nonlinearity function is a polynomial we simply cut off the series. It is possible to choose the nonlinearity function in different ways but it is necessary that some conditions (it will be written below) are satisfied[1].

For the values $\tau(0)$ and $\tau(h)$ we obtain

$$\tau(0) = \varepsilon_2 + Y_0^2, \quad \tau(h) = \varepsilon_2 + Y_h^2;$$

since Y_h is known, so is $\tau(h)$.

In accordance with the transmission conditions for $\eta(0)$ and $\eta(h)$ we have

$$\eta(0) = \sqrt{\gamma^2 - \varepsilon_1} > 0, \quad \eta(h) = -\sqrt{\gamma^2 - \varepsilon_3} < 0. \tag{20}$$

From first integral (19), at $x = h$, we find $C_h^\tau := C|_{x=h}$

$$C_h^\tau = (\varepsilon_2 - \varepsilon_3)(\tau(h) - \varepsilon_2) + \varphi(\tau(h) - \varepsilon_2). \tag{21}$$

Now from first integral (19), using (20) and (21), we find equation with respect to $\tau(0)$:

$$(\varepsilon_2 - \varepsilon_1)(\tau(0) - \varepsilon_2) + \varphi(\tau(0) - \varepsilon_2) = \\ = (\varepsilon_2 - \varepsilon_3)(\tau(h) - \varepsilon_2) + \varphi(\tau(h) - \varepsilon_2). \tag{22}$$

It is obvious that $\tau(0) \geq \varepsilon_2$ since $\tau(0) = \varepsilon_2 + Y_0^2$ and $\varepsilon_2 > 0$. For existing the root $\tau(0) \geq \varepsilon_2$ of equation (22) it is necessary to impose some conditions on the function f. For example, if f is a polynomial with nonnegative coefficients, then the suitable root exists.

[1] Different types of nonlinearity functions (that differ from polynomial) are shown in [4].

It should be noticed that from equation (22) it can be seen that if $\varepsilon_1 = \varepsilon_3$, then one of the roots of this equation is $\tau(h)$, i.e., $\tau(0) = \tau(h)$. In original variables we obtain $Y_0^2 = Y_h^2$. The situation is almost the same for the case of a linear layer (see Ch. 2). There is a slight difference here between the case of a nonlinear layer and the case of a linear layer. In the linear case it is always $Y_0^2 = Y_h^2$ when $\varepsilon_1 = \varepsilon_3$. In the nonlinear case this is only one root of equation (22).

We suppose that the function f satisfies the condition

$$\gamma^2 - \varepsilon_2 - f(\tau - \varepsilon_2) - \eta^2 < 0.$$

It is surely true if f is a polynomial with nonnegative coefficients.

In this case right-hand side of the second equation of system (18) is negative. This means that the function η decreases for $x \in (0, h)$. From formulas (20) we can see that $\eta(0) > 0$ and $\eta(h) < 0$. However, it is possible that there are zeros of the function Y. Since Y and Z are analytical functions, so is η. This means that η has discontinuities of the second kind at the points x^*, where $Y(x^*) = 0$. These points are poles of the function η.

From first integral (19) we have

$$\eta^2 = \frac{C_h^\tau - \varphi(\tau - \varepsilon_2) - (\varepsilon_2 - \gamma^2)(\tau - \varepsilon_2)}{\tau - \varepsilon_2}.$$

.

The poles are zeros of the denominator of this expression. Then, in these points $\tau^* = \tau(x^*)$ is such that $\eta^* = \pm\infty$.

Let us suppose that there are $(N+1)$ points of discontinuities: x_0, ..., x_N on the interval $(0, h)$.

The properties of function $\eta = \eta(x)$ imply

$$\eta\,(x_i - 0) = -\infty, \quad \eta\,(x_i + 0) = +\infty, \quad \text{where} \quad i = \overline{0, N}. \tag{23}$$

Denote by

$$w := \left[\gamma^2 - \varepsilon_2 - f(\tau - \varepsilon_2) - \eta^2\right]^{-1},$$

where $w \equiv w(\eta)$; and $\tau = \tau(\eta)$ is expressed from first integral (19).

Taking into account our hypothesis we will seek to the solutions on each interval $[0\ ,\ x_0)$, (x_0, x_1), ..., $(x_N\ ,\ h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0)} wd\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} wd\eta = x + c_i, & x_i \le x \le x_{i+1},\ \ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} wd\eta = x + c_N, & x_N \le x \le h. \end{cases} \tag{24}$$

Substituting $x = 0$, $x = x_{i+1}$, and $x = x_N$ into equations (24) (into the first, the second, and the third, respectively) and taking into account (23), we find constants $c_1,\ c_2,\ ...,\ c_{N+1}$:

$$\begin{cases} c_0 = -\int\limits_{\eta(0)}^{-\infty} wd\eta; \\ c_{i+1} = \int\limits_{+\infty}^{-\infty} wd\eta - x_{i+1},\ \ i = \overline{0, N-1}; \\ c_{N+1} = \int\limits_{+\infty}^{\eta(h)} wd\eta - h. \end{cases} \tag{25}$$

Using (25) we can rewrite equations (24) in the following form

$$\begin{cases} \int\limits_{\eta(x)}^{\eta(x_0)} wd\eta = -x + \int\limits_{\eta(0)}^{-\infty} wd\eta, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} wd\eta = x + \int\limits_{+\infty}^{-\infty} wd\eta - x_{i+1}, & x_i \le x \le x_{i+1},\ \ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} wd\eta = x + \int\limits_{+\infty}^{\eta(h)} wd\eta - h, & x_N \le x \le h. \end{cases} \tag{26}$$

Introduce the notation $T := -\int\limits_{-\infty}^{+\infty} \omega d\eta$. It follows from formula (26) that $0 < x_{i+1} - x_i = T < h$, where $i = \overline{0, N-1}$. This implies the convergence of the improper integral (it will be proved in other way below). Now setting x in equations (26) such that all the integrals on the left side vanish (i.e., $x = x_0$, $x = x_i$, and $x = x_N$), and sum all equations (26). We obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{-\infty} w d\eta + x_0 + T - x_1 + \dots$$

$$\dots + x_{N-1} + T - x_N + x_N + \int\limits_{+\infty}^{\eta(h)} w d\eta - h.$$

Finally we have

$$-\int\limits_{-\sqrt{\gamma^2-\varepsilon_3}}^{\sqrt{\gamma^2-\varepsilon_1}} w d\eta + (N+1)\,T = h. \qquad (27)$$

Expression (27) is the DE, which holds for any finite h. Let γ be a solution of DE (27) and an eigenvalue of the problem. Then, there is an eigenfunction Y, which corresponds to the eigenvalue γ. The eigenfunction Y has $N+1$ zeros on the interval $(0, h)$.

Notice that the improper integrals in DE (27) converge. Indeed, function $\tau = \tau(\eta)$ is bounded as $\eta \to \infty$ since $\tau = \varepsilon_2 + Y^2$ and Y is bounded. Then

$$|w| = \left| \frac{1}{\gamma^2 - \varepsilon_2 - f(\tau - \varepsilon_2) - \eta^2} \right| \le \left| \frac{1}{\eta^2 + \alpha} \right|,$$

where $\alpha > 0$ is a constant. It is obvious that improper integral $\int\limits_{-\infty}^{+\infty} \frac{d\eta}{\eta^2+\alpha}$ converges. Convergence of the improper integrals in (27) in inner points results from the requirement that right-hand side of the second equation of system (18) is negative.

Theorem 1. *The set of solutions of DE* (27) *contains the set of solutions* (*eigenvalues*) *of the boundary eigenvalue problem* (12)–(14) *with conditions* (10), (11), (15).

Proof. It follows from the method of obtaining of DE (27) from system (18) that an eigenvalue of the problem (12)–(14) is a solution of the DE.

It is obvious that the function τ as the function with respect to η defined from first integral (19) is a multiple-valued function. This implies that not every solution of DE (27) is an eigenvalue of the problem. In other words, equation (22) can have several roots $\tau(0)$ such that each of them satisfies the condition $\tau(0) \geq \varepsilon_2$. Even in this case it is possible to find eigenvalues among roots of the DE. Indeed, when we find a solution γ of DE (27), we can find functions $\tau(x)$ and $\eta(x)$ from system (18) and first integral (19). From functions $\tau(x)$ and $\eta(x)$, using formulas (16), (17) we obtain

$$Y(x) = \pm\sqrt{\tau - \varepsilon_2} \quad \text{and} \quad Z(x) = \pm\sqrt{\tau - \varepsilon_2}|\eta|. \tag{28}$$

It is an important question how to choose the sign. Let us discuss it in detail. We know that the function η monotonically decreases. If $x = x^*$ such that $\eta\,(x^*) = 0$, then $\eta\,(x^* - 0) > 0$, $\eta\,(x^* + 0) < 0$; and if $x = x^{**}$ such that $\eta\,(x^{**}) = \pm\infty$, then $\eta\,(x^{**} - 0) < 0$ and $\eta\,(x^{**} + 0) > 0$. The function η has no other points of sign's reversal. To fix the idea, assume that the initial condition is $Y_h > 0$. If $\eta > 0$, then the functions Y and Z have the same signs; and if $\eta < 0$, then Y and Z have different signs. Since Y and Z are continuous we can choose necessary signs in expressions (28). Now, when we found the function Y we can calculate $\tau(0) = \varepsilon_2 + Y_0^2$. If this calculated value is equal to the value calculated from equation (22), then the solution γ of the DE is an eigenvalue of the problem (and is not an eigenvalue otherwise).

If function φ such that a unique root $\tau(0) \geq \varepsilon_2$ of equation (22) exists, then we have the following

Theorem 2 (*of equivalence*). *If equation* (22) *has a unique solution* $\tau(0) \geq \varepsilon_2$, *then boundary eigenvalue problem* (12)–(14) *with conditions* (10), (11), (15) *has a solution* (*an eigenvalue*) *if and only if this eigenvalue is a solution of DE* (27).

The proof of this theorem results from the proof of previous theorem.

Introduce the notation $J(\gamma, k) := -\int\limits_{\eta(0)}^{\eta(h)} w d\eta + kT$, where the right-hand side is defined by DE (27) and $k = \overline{0, N+1}$.

Let

$$h^k_{\inf} = \inf_{\gamma^2 \in (\max(\varepsilon_1, \varepsilon_3), \varepsilon_2)} J(\gamma, k),$$
$$h^k_{\sup} = \sup_{\gamma^2 \in (\max(\varepsilon_1, \varepsilon_3), \varepsilon_2)} J(\gamma, k).$$

Let us formulate the sufficient condition of existence at least one eigenvalue of the problem.

Theorem 3. *Let h satisfies for a certain $k = \overline{0, N+1}$ the following two-sided inequality*

$$h^k_{\inf} < h < h^k_{\sup},$$

then boundary eigenvalue problem (12)–(14) *with conditions* (10), (11), (15) *has at least one solution* (*an eigenvalue*).

The quantities $h^k_{\inf}$ and $h^k_{\sup}$ can be numerically calculated.

§6. Generalized Dispersion Equation

Here we derive the generalized DE which holds for any real values ε_2. In addition the sign of the right-hand side of the second equation in system (18)[1], and conditions $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$ or $0 < \gamma^2 < \varepsilon_2$ will not be taken into account. These conditions appear in the case of a linear layer and are used for derivation of DE (27). Though in the nonlinear case it is not necessary to limit the value γ from the right side. At the same time it is clear that γ is limited from the left side since this limit appears from the solutions in the half-spaces (where the permittivities are constants).

[1]Such a condition appears in the analogous problem for TM waves propagating through a layer with Kerr nonlinearity (see Ch. 7). In Ch. 7 this condition naturally results from the problem in a linear layer. Of course, it would be possible to derive generalized DE at once and § 5 can be omitted. However, when the right-hand side preserves its sign the way of DEs' derivation is quite transparent.

Now we assume that γ satisfies one of the following inequalities

$$\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < +\infty,$$

when either ε_1 or ε_3 is positive, or

$$0 < \gamma^2 < +\infty,$$

when both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$.

At first we derive the DE from system (18) and first integral (19). After this we discuss the details of the derivation and conditions when the derivation is possible and the DE is well defined.

Using first integral (19) it is possible to integrate formally any of the equations of system (18). As earlier we integrate the second equation. We can not obtain a solution on the whole interval $(0, h)$ since the function $\eta(x)$ can have break points which belong to $(0, h)$. It is known that the function $\eta(x)$ is an analytical one. Therefore we can conclude that if $\eta(x)$ has break points when $x \in (0,\, h)$, then there are only break points of the second kind.

Assume that the function $\eta(x)$ on the interval $(0, h)$ has $N+1$ break points $x_0,\ x_1,\ ...,\ x_N$.

It should be noticed that

$$\eta(x_i - 0) = \pm\infty, \quad \eta(x_i + 0) = \pm\infty,$$

where $i = \overline{0,\, N}$, and signs $\pm$ in these formulas are independent and unknown.

Taking into account the above, solutions are sought on each of the intervals $[0\ , x_0)$, (x_0, x_1), ..., $(x_N\ , h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + c_{i+1}, & x_i \le x \le x_{i+1},\ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + c_{N+1}, & x_N \le x \le h. \end{cases} \qquad (29)$$

From equation (29), substituting $x = 0$, $x = x_{i+1}$, $x = x_N$ into the first, the second, and the third equations (29), respectively, we find required constants c_1, c_2, ..., c_{N+1}:

$$\begin{cases} c_0 = -\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta; \\ c_{i+1} = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, \ i = \overline{0, N-1}; \\ c_{N+1} = \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h. \end{cases} \tag{30}$$

Using (30) equations (29) take the form

$$\begin{cases} \int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = -x + \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, & x_i \le x \le x_{i+1}; \\ \int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h, & x_N \le x \le h, \end{cases} \tag{31}$$

where $i = \overline{0, N-1}$.

From formulas (31) we obtain that

$$x_{i+1} - x_i = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta, \tag{32}$$

where $i = \overline{0, N-1}$.

Expressions $0 < x_{i+1} - x_i < h < \infty$ imply that under the assumption about the break points existence the integral on the right side converges and $\int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta > 0$.

In the same way, from the first and the last equations (31) we obtain that $x_0 = \int\limits_{\eta(0)}^{\eta(x_0-0)} wd\eta$ and $0 < x_0 < h$ then

$$0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} wd\eta < h < \infty;$$

and $h - x_N = \int\limits_{\eta(x_N+0)}^{\eta(h)} wd\eta$ and $0 < h - x_N < h$ then

$$0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} wd\eta < h < \infty.$$

These considerations yield that the function $w(\eta)$ has no non-integrable singularities for $\eta \in (-\infty, \infty)$. And also this proves that the assumption about a finite number break points is true.

Now, setting $x = x_0$, $x = x_i$, and $x = x_N$ in the first, the second, and the third equations in (31), respectively, we have that all the integrals on the left-sides vanish. We add all the equations in (31) to obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{\eta(x_0-0)} wd\eta + x_0 + \int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} wd\eta - x_1 + ...$$
$$... + x_{N-1} + \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} wd\eta - x_N + x_N + \int\limits_{\eta(x_N+0)}^{\eta(h)} wd\eta - h. \quad (33)$$

From (33) we obtain

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} wd\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} wd\eta + \sum_{i=0}^{N-1} \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} wd\eta = h. \quad (34)$$

It follows from formulas (32) that

$$\eta(x_i + 0) = \pm\infty \text{ and } \eta(x_i - 0) = \mp\infty,$$

where $i = \overline{0, N}$, and it is necessary to choose the infinities of different signs.

Thus we obtain that

$$\int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} w d\eta = \ldots = \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} w d\eta =: T'.$$

Hence $x_1 - x_0 = \ldots = x_N - x_{N-1}$.

Now we can rewrite equation (34) in the following form

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} f d\eta + NT' = h.$$

Let $T := -\int\limits_{-\infty}^{+\infty} w d\eta$, then we finally obtain

$$-\int\limits_{-\sqrt{\gamma^2-\varepsilon_3}}^{\sqrt{\gamma^2-\varepsilon_1}} w d\eta \pm (N+1)T = h. \tag{35}$$

Expression (35) is the DE, which holds for any finite h. Let γ be a solution of DE (35) and an eigenvalue of the problem. Then, there is an eigenfunction Y, which corresponds to the eigenvalue γ. The eigenfunction Y has $N+1$ zeros on the interval $(0, h)$. It should be noticed that for every number $N+1$ it is necessary to solve two DEs: for $N+1$ and for $-(N+1)$.

Let us formulate the following

Theorem 4. *The set of solutions of DE* (35) *contains the set of solutions* (*eigenvalues*) *of boundary eigenvalues problem* (12)–(14) *with conditions* (10), (11), (15).

The proof of this theorem is almost word-by-word coincides to the proof of Theorem 1.

Now let us review some theoretical treatments of the derivation of DEs (27) and (35). We are going to discuss the existence and uniqueness of system's (8) solution.

Let us consider vector form (13) of system (8):

$$\mathrm{D}\mathbf{F} = \mathbf{G}(\mathbf{F}, \lambda). \tag{36}$$

Let the right-hand side $\mathbf{G}$ be defined and continuous in the domain $\Omega \subset \mathbb{R}^2$, $\mathbf{G} : \Omega \to \mathbb{R}^2$. Also we suppose that $\mathbf{G}$ satisfies the Lipschitz condition on $\mathbf{F}$ (locally in Ω)[1].

Under these conditions system (8) (or system (36)) has a unique solution in the domain Ω [8, 41, 22].

It is clear that under these conditions system (18) has a unique solution (of course, the domain of uniqueness Ω' for variables τ, η differs from Ω).

Since we seek bounded solutions Y and Z; therefore,

$$\Omega \subset [-m_1, m_1] \times [-m_2, m_2],$$

where

$$\max_{x \in [0,h]} |Y| < m_1, \quad \max_{x \in [0,h]} |Z| < m_2,$$

and the previous implies that

$$\Omega' \subset [\varepsilon_2, \varepsilon_2 + m_1^2] \times (-\infty, +\infty).$$

[1] Let $x \in \mathbb{R}^2$, Ω be a domain in $\mathbb{R}^2$ and G is a continuous function of two variables.

Function $G : \Omega \to \mathbb{R}^2$ satisfies the Lipschitz condition on x (globally in Ω) if

$$\overline{x}, \overline{\overline{x}} \in \Omega \ \Rightarrow \ ||G(\overline{x}) - G(\overline{\overline{x}})|| \leq L||\overline{\overline{x}} - \overline{x}||,$$

where $L > 0$ is a constant which does not depend on points $\overline{x}$ and $\overline{\overline{x}}$ (the Lipschitz constant).

Function $G : \Omega \to \mathbb{R}^2$ satisfies the Lipschitz condition on x locally in Ω, if for any point $x_0 \in \Omega$ exists its neighborhood $V(x_0)$, then the section of the function G in $V(x_0)$ satisfies the Lipschitz condition globally in $V(x_0)$.

Under our assumptions the right-hand side of system (36) is analytical and, therefore, the Lipschitz condition is fulfilled. This means that for such a system all mentioned statements about existence and uniqueness of a solution holds.

It is easy to show that there is no point $x^* \in \Omega'$, such that $Y|_{x=x^*} = 0$ and $Z|_{x=x^*} = 0$. Indeed, it is known from theory of autonomous system (see for example [41]) that phase trajectories do not intersect one another in the system's phase space when the right-hand side of the system is continuous and satisfies the Lipschitz condition. Since $Y \equiv 0$ and $Z \equiv 0$ are stationary solutions of system (8), it is obvious that nonconstant solutions Y and Z can not vanish simultaneously in a certain point $x^* \in \Omega'$ (otherwise the nonconstant solutions intersect with the stationary solutions and we obtain a contradiction).

Note 1. If there is a certain value γ_*^2 such that some of the integrals in DEs (27) or (35) diverge in certain inner points, then this simply means that the value γ_*^2 is not a solution of chosen DE and the value γ_*^2 is not an eigenvalue of the problem.

Note 2. It is necessary to emphasize that this boundary eigenvalue problem essentially depends on the initial condition Y_h. The transmission problem for a linear layer does not depend on the initial condition. If the nonlinearity function is a specific one, then in some cases it will be possible to normalize the Maxwell equations in such way that the transmission problem does not depend on initial condition Y_h explicitly (it is possible for example for Kerr nonlinearity in layers and in circle cylindrical waveguides). Stress the fact once more that the opportunity of such normalization is an exceptional case. What is more, in spite of the fact that this normalization is possible in certain cases it does not mean that the normalized transmission problem is independent of the initial condition. In this case one of the problem's parameter depends on the initial conditions. The way of such a normalization see in Ch. 11, 13.

We derived DEs from the second equation of system (18). It is possible to do it using the first equation of the system (see p. 130).

§7. Numerical Results

Let us consider the following nonlinearity (see (5))

$$f = aY^2 + bY^4 + cY^6 + dY^8,$$

where a, b, c, and d are arbitrary real constants.

Using DE (35) we calculate dispersion curves (DC). The following parameters $\varepsilon_1 = \varepsilon_3 = 1$, $\varepsilon_2 = 3$, $Y_h = 1$ are used.

In Fig. 2,a,b the behavior of dispersion curves is shown. The dashed curves are DCs for the linear layer (when $f \equiv 0$), the lines $\gamma^2 = 3$ are asymptotes for DCs in the linear case, solid curves are the DCs for the nonlinear case (solutions of DE (35)).

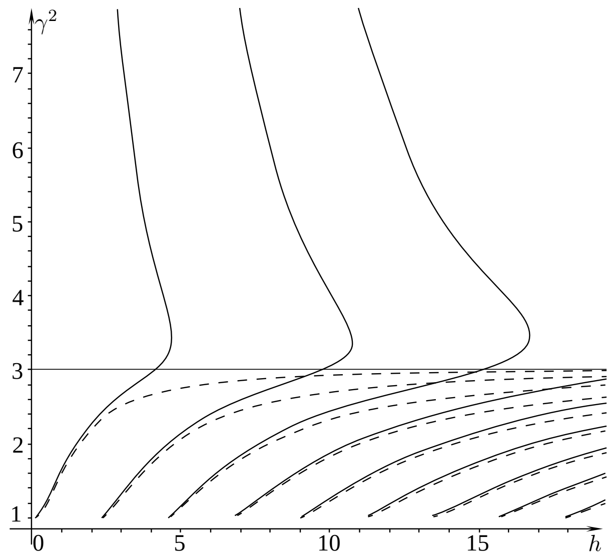


Figure 2,a. $a = b = c = d = 0.05$

It follows from numerical calculations (for the case of Kerr nonlinearity it can be proved strictly) that the function $h \equiv h(\gamma)$ defined from equation (35) when $f > 0$ has the following property: $\lim\limits_{\gamma^2 \to +\infty} h(\gamma) = 0$. This means that in the nonlinear case every DC has an asymptote, and the asymptote is $h = 0$. How are the value of the propagation constant and its number defined? For example, see

Fig. 2,b. The line $h = 13$ corresponds to the layer's thickness. For the linear layer in this case there are 6 propagation constants (black dots where the line $h = 13$ intersects the dashed DCs). These propagation constants are eigenvalues of the problem correspond to the eigenmodes. In the case of the nonlinear layer in Fig. 2,b are shown 4 eigenvalues (uncolored dots). These eigenvalues corresponds to 4 eigenmodes. Taking the above into account it is easy to understand that in the nonlinear case there are infinite number of eigenvalues.

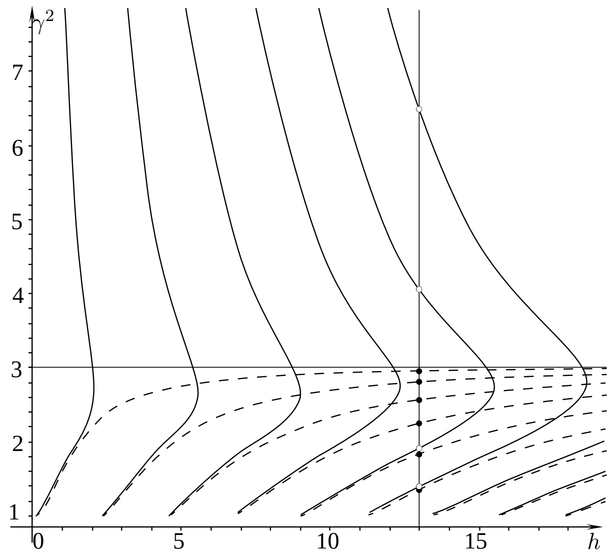


Figure 2,b. $a = 0.05$, $b = c = d = 0.005$

CHAPTER 4

TE WAVE PROPAGATION IN A LAYER WITH GENERALIZED KERR NONLINEARITY

§1. Statement of the Problem

Let us consider electromagnetic waves propagating through a homogeneous isotropic nonmagnetic dielectric layer. The layer is located between two half-spaces: $x < 0$ and $x > h$ in Cartesian coordinate system $Oxyz$. The half-spaces are filled with isotropic nonmagnetic media without any sources and characterized by permittivities $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, respectively, where ε_0 is the permittivity of free space[1]. Assume that everywhere $\mu = \mu_0$, where μ_0 is the permeability of free space.

The electromagnetic field depends on time harmonically [17]

$$\tilde{\mathbf{E}}(x, y, z, t) = \mathbf{E}_+(x, y, z)\cos\omega t + \mathbf{E}_-(x, y, z)\sin\omega t,$$
$$\tilde{\mathbf{H}}(x, y, z, t) = \mathbf{H}_+(x, y, z)\cos\omega t + \mathbf{H}_-(x, y, z)\sin\omega t,$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_+,\ \mathbf{E}_-,\ \tilde{\mathbf{H}},\ \mathbf{H}_+,\ \mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = (E_x, E_y, E_z)^T, \quad \mathbf{H} = (H_x, H_y, H_z)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

[1]Generally, conditions $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$ are not necessary. They are not used for derivation of DEs, but they are useful for DEs' solvability analysis.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential field components on the media interfaces $x = 0$, $x = h$ and the radiation condition at infinity: the electromagnetic field exponentially decays as $|x| \to \infty$ in the domains $x < 0$ and $x > h$.

The permittivity inside the layer has the form[1]

$$\varepsilon = \varepsilon_2 + a|\mathbf{E}|^2 + b|\mathbf{E}|^4,$$

where ε_2 is the constant part of the permittivity; $a > 0$, $b > 0$ are the nonlinearity coefficients. We assume that $\varepsilon_2 > \max(\varepsilon_1, \varepsilon_3)$. In §6 we assume that ε_2, a, b are arbitrary real constants.

The solutions to the Maxwell equations are sought in the entire space.

The geometry of the problem is shown in Fig. 1.

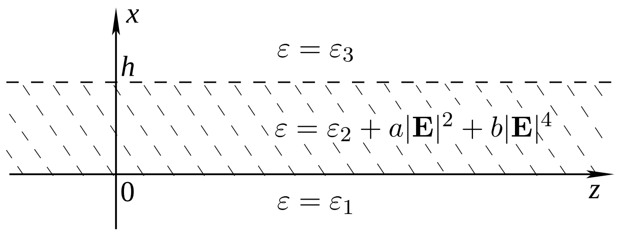


Fig. 1.

§2. TE Waves

Let us consider TE waves

$$\mathbf{E} = (0, E_y, 0)^T, \quad \mathbf{H} = (H_x, 0, H_z)^T,$$

where $E_y = E_y(x, y, z)$, $H_x = H_x(x, y, z)$, and $H_z = H_z(x, y, z)$.

[1]Such a nonlinearity is called generalized Kerr nonlinearity; if $b = 0$, then we obtain Kerr nonlinearity; if $a = b = 0$, then we obtain the linear case studied in Ch. 2.

Substituting the fields into Maxwell equations (1) we obtain

$$\begin{cases} \frac{\partial H_z}{\partial y} = 0, \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon E_y, \\ \frac{\partial H_x}{\partial y} = 0, \\ \frac{\partial E_y}{\partial z} = -i\omega\mu H_x, \\ \frac{\partial E_y}{\partial x} = i\omega\mu H_z. \end{cases}$$

It is obvious from the first and the third equations of this system that H_z and H_x do not depend on y. This implies that E_y does not depend on y.

Waves propagating along medium interface z depend on z harmonically. This means that the fields components have the form

$$E_y = E_y(x)e^{i\gamma z}, H_x = H_x(x)e^{i\gamma z}, H_z = H_z(x)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system

$$\begin{cases} i\gamma H_x(x) - H_z'(x) = -i\omega\varepsilon E_y(x), \\ -i\gamma E_y(x) = i\omega\mu H_x(x), \\ E_y'(x) = i\omega\mu H_z(x), \end{cases} \tag{2}$$

where $(\,\cdot\,)' \equiv \frac{d}{dx}$.

After simple transformation of system (2) we obtain

$$\gamma^2 E_y\left(x\right) - E_y''\left(x\right) = \omega^2\mu\varepsilon E_y\left(x\right).$$

Let us denote by $k_0^2 := \omega^2\mu_0\varepsilon_0$, and perform the normalization according to the formulas $\tilde{x} = k_0 x$, $\frac{d}{dx} = k_0\frac{d}{d\tilde{x}}$, $\tilde{\gamma} = \frac{\gamma}{k_0}$, $\tilde{\varepsilon}_j = \frac{\varepsilon_j}{\varepsilon_0}$ $(j = 1, 2, 3)$, $\tilde{a} = \frac{a}{\varepsilon_0}$, $\tilde{b} = \frac{b}{\varepsilon_0}$. Denoting by $Y(\tilde{x}) := E_y(\tilde{x})$ and omitting the tilde symbol, we have

$$Y''\left(x\right) = \gamma^2 Y\left(x\right) - \varepsilon Y\left(x\right). \tag{3}$$

Introducing the function $Z(x) := Y'(x)$ we can consider (3) as the following system of equations

$$\begin{cases} Y'(x) = Z(x), \\ Z'(x) = \left(\gamma^2 - \varepsilon\right) Y(x). \end{cases} \tag{4}$$

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along boundaries of the layer $0 < x < h$, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of spectral parameter γ such that real solutions $Y(x)$ and $Z(x)$ to system (4) exist (see the note on p. 23 and the footnote on p. 33). We assume that

$$\varepsilon = \begin{cases} \varepsilon_1, & x < 0; \\ \varepsilon_2 + aY^2 + bY^4, & 0 < x < h; \\ \varepsilon_3, & x > h. \end{cases} \tag{5}$$

Also we assume that functions Y and Z are sufficiently smooth

$$Y(x) \in C(-\infty, +\infty) \cap$$
$$\cap\, C^1(-\infty, +\infty) \cap C^2(-\infty, 0) \cap C^2(0, h) \cap C^2(h, +\infty),$$
$$Z(x) \in C(-\infty, +\infty) \cap C^1(-\infty, 0) \cap C^1(0, h) \cap C^1(h, +\infty).$$

Physical nature of the problem implies these conditions.

It is clear that system (4) is an autonomous one with analytical right-hand sides with respect to Y and Z. It is well known (see [5]) that solutions Y and Z of such a system are also analytical functions with respect to independent variable. This is an important fact for DEs' derivation.

We will seek γ under condition $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$.

This condition corresponds to the classical problem of TE wave propagating in a linear layer, when $\varepsilon_1 \geq \varepsilon_0$, $\varepsilon_3 \geq \varepsilon_0$, and ε in the layer is equal to ε_2 and $\varepsilon_2 > \max(\varepsilon_1, \varepsilon_3)$. This condition naturally appears in that problem (see. Ch. 2), therefore we use it to derive the DEs for a nonlinear layer. In §6 the DE is obtained under the most general conditions. We also notice that condition $\gamma^2 > \max(\varepsilon_1, \varepsilon_3)$ holds if at least one of the values ε_1 or ε_3 more than zero. If both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem

In the domain $x < 0$ we have $\varepsilon = \varepsilon_1$. From system (4) we obtain $Y'' = \left(\gamma^2 - \varepsilon_1\right) Y$. Its general solution is

$$Y(x) = A_1 e^{-x\sqrt{\gamma^2-\varepsilon_1}} + Ae^{x\sqrt{\gamma^2-\varepsilon_1}}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} Y(x) &= A \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right), \\ Z(x) &= A\sqrt{\gamma^2 - \varepsilon_1} \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right). \end{aligned} \tag{6}$$

We assume that $\gamma^2 - \varepsilon_1 > 0$ otherwise it will be impossible to satisfy the radiation condition.

In the domain $x > h$ we have $\varepsilon = \varepsilon_3$. From system (4) we obtain $Y'' = \left(\gamma^2 - \varepsilon_3\right) Y$. Its general solution is

$$Y(x) = B_1 e^{(x-h)\sqrt{\gamma^2-\varepsilon_3}} + Be^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} Y(x) &= B \exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right), \\ Z(x) &= -\sqrt{\gamma^2 - \varepsilon_3} B \exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right). \end{aligned} \tag{7}$$

Her for the same reason as above we consider that $\gamma^2 - \varepsilon_3 > 0$.

Constants A and B in (6) and (7) are defined by transmission conditions and initial conditions.

Inside the layer $0 < x < h$ system (4) takes the form

$$\begin{cases} Y'(x) = Z(x), \\ Z'(x) = (\gamma^2 - \varepsilon_2 - aY^2(x) - bY^4(x))Y(x). \end{cases} \tag{8}$$

System (8) has the first integral. So we can study the first-order equation (either the first or the second one in (8)) with the first integral instead of second-order equation (3). Divide the second equation in (8) by the other one we obtain

$$ZdZ + (\varepsilon_2 - \gamma^2 + aY^2 + bY^4)YdY = 0.$$

This equation is the total differential equation. Its general solution has the form

$$6Z^2 + 6(\varepsilon_2 - \gamma^2)Y^2 + 3aY^4 + 2bY^6 \equiv C, \tag{9}$$

where C is a constant of integration.

§4. Transmission Conditions and the Transmission Problem

Tangential components of an electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are E_y and H_z. Hence we obtain

$$E_y(h+0) = E_y(h-0), \quad E_y(0-0) = E_y(0+0),$$
$$H_z(h+0) = H_z(h-0), \quad H_z(0-0) = H_z(0+0).$$

The continuity conditions for components E_y, H_z and formulas (2), (4) imply the transmission conditions for Y, Z

$$[Y]_{x=0} = 0, \quad [Y]_{x=h} = 0, \quad [Z]_{x=0} = 0, \quad [Z]_{x=h} = 0, \tag{10}$$

where $[f]_{x=x_0} = \lim\limits_{x \to x_0 - 0} f(x) - \lim\limits_{x \to x_0 + 0} f(x)$ denotes a jump of the function f at the interface.

Denote by $Y_0 := Y(0-0)$, $Y_h := Y(h+0)$, $Z_0 := Z(0-0)$, and $Z_h := Z(h+0)$. Then, we obtain $A = Y_0$, $B = Y_h$ and

$$Z_0 = \sqrt{\gamma^2 - \varepsilon_1}\,Y_0, \quad Z_h = -\sqrt{\gamma^2 - \varepsilon_3}\,Y_h.$$

The constant Y_h is supposed to be known (initial condition).

We also suppose that functions $Y(x)$, $Z(x)$ satisfy the condition

$$Y(x) = O\left(\frac{1}{|x|}\right), \; Z(x) = O\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty. \tag{11}$$

Let

$$\mathrm{D} = \begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & \frac{d}{dx} \end{pmatrix}, \mathbf{F}(X,Z) = \begin{pmatrix} X \\ Z \end{pmatrix}, \mathbf{G}(\mathbf{F},\gamma) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

where $Y \equiv Y(x)$, $Z \equiv Z(x)$ are unknown functions, $G_1 \equiv G_1(\mathbf{F},\gamma)$, $G_2 \equiv G_2(\mathbf{F},\gamma)$ are right-hand sides of system (8). The value γ is a spectral parameter. Rewrite the problem using new notation.

For the half-space $x < 0$ and $\varepsilon = \varepsilon_1$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & 1 \\ \gamma^2 - \varepsilon_1 & 0 \end{pmatrix} \mathbf{F} = 0. \tag{12}$$

Inside the layer $0 < x < h$ and $\varepsilon = \varepsilon_2 + aY^2 + bY^4$ we have

$$\mathrm{L}(\mathbf{F},\gamma) \equiv \mathrm{D}\mathbf{F} - \mathbf{G}(\mathbf{F},\gamma) = 0. \tag{13}$$

For the half-space $x > h$ and $\varepsilon = \varepsilon_3$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & 1 \\ \gamma^2 - \varepsilon_3 & 0 \end{pmatrix} \mathbf{F} = 0. \tag{14}$$

Substituting the value Y_h into first integral (9) we find the integration constant value $C_h^Y := C|_{x=h}$

$$6Z_h^2 + 6(\varepsilon_2 - \gamma^2)Y_h^2 + 3aY_h^4 + 2bY_h^6 = C_h^Y.$$

Taking into account that $Z_h = -\sqrt{\gamma^2 - \varepsilon_3}\,Y_h$ we finally obtain

$$C_h^Y = 6(\varepsilon_2 - \varepsilon_3)Y_h^2 + 3aY_h^4 + 2bY_h^6.$$

First integral (9), the value C_h^Y, and the value $Z_0 = \sqrt{\gamma^2 - \varepsilon_1}\,Y_0$ imply the equation with respect to Y_0^2:

$$6(\varepsilon_2 - \varepsilon_1)Y_0^2 + 3aY_0^4 + 2bY_0^6 = 6(\varepsilon_2 - \varepsilon_3)Y_h^2 + 3aY_h^4 + 2bY_h^6. \tag{15}$$

It is easy to see from equation (15) that under condition $\varepsilon_1 = \varepsilon_3$ it has at least one root Y_h^2, i.e., $Y_0^2 = Y_h^2$. It should be noticed that in a linear case we always have $Y_0^2 = Y_h^2$. In the nonlinear case it is only one of the roots.

Let us formulate the transmission problem (it is possible to reformulate it as the boundary eigenvalue problem). *It is necessary to find eigenvalues* γ *and corresponding to them nonzero vectors* $\mathbf{F}$ *such that* $\mathbf{F}$ *satisfies to equations* (12)–(14). *Components* Y, Z *of vector* $\mathbf{F}$ *satisfy transmission conditions* (10), *condition* (11), *and* $Y(0) \equiv Y_0$ *is defined from equation* (15).

Definition 1. The value $\gamma = \gamma_0$ such that nonzero solution $\mathbf{F}$ to problem (12)–(14) exists under conditions (10), (11), and (15) is called an eigenvalue of the problem. Solution $\mathbf{F}$, corresponding to the eigenvalue is called an eigevector of the problem, and components $Y(x)$ and $Z(x)$ of vector $\mathbf{F}$ are called eigenfunctions (see the note on p. 37).

§5. Dispersion Equation

Introduce the new variables

$$\tau(x) = \varepsilon_2 + Y^2(x), \quad \eta(x) = \frac{Z(x)}{Y(x)}, \tag{16}$$

from (16) we obtain

$$Y^2 = \tau - \varepsilon_2, \quad YZ = (\tau - \varepsilon_2)\eta, \quad Z^2 = (\tau - \varepsilon_2)\eta^2. \tag{17}$$

System (8) takes the form

$$\begin{cases} \tau' = 2(\tau - \varepsilon_2)\eta, \\ \eta' = \gamma^2 - \varepsilon_2 - a(\tau - \varepsilon_2) - b(\tau - \varepsilon_2)^2 - \eta^2. \end{cases} \tag{18}$$

First integral (9) has the form

$$\eta^2 = \frac{C - 6(\varepsilon_2 - \gamma^2)(\tau - \varepsilon_2) - 3a(\tau - \varepsilon_2)^2 - 2b(\tau - \varepsilon_2)^3}{6(\tau - \varepsilon_2)}. \tag{19}$$

It is easy to see from first integral (19) that there are algebraic dependence between functions τ and η. The function $\tau = \tau(\eta)$ is expressed from (19) by the Cardanus formulas [28].

From formulas (16) we obtain $\tau(0) = \varepsilon_2 + Y_0^2$, $\tau(h) = \varepsilon_2 + Y_h^2$; since Y_h is known, so is $\tau(h)$.

In accordance with the transmission conditions (10) for $\eta(0)$ and $\eta(h)$ we have

$$\eta(0) = \sqrt{\gamma^2 - \varepsilon_1} > 0, \quad \eta(h) = -\sqrt{\gamma^2 - \varepsilon_3} < 0. \tag{20}$$

From first integral (19), at $x = h$, we find $C_h^\tau := C|_{x=h}$

$$C_h^\tau = 6(\varepsilon_2 - \varepsilon_3)(\tau(h) - \varepsilon_2) + 3a(\tau(h) - \varepsilon_2)^2 + 2b(\tau(h) - \varepsilon_2)^3. \tag{21}$$

Now from first integral (19), using (20) and (21), we find the equation with respect to $\tau(0)$:

$$\begin{aligned} 6(\varepsilon_2 - \varepsilon_1)(\tau(0) - \varepsilon_2) + 3a(\tau(0) - \varepsilon_2)^2 + 2b(\tau(0) - \varepsilon_2)^3 = \\ = 6(\varepsilon_2 - \varepsilon_3)(\tau(h) - \varepsilon_2) + 3a(\tau(h) - \varepsilon_2)^2 + 2b(\tau(h) - \varepsilon_2)^3. \end{aligned} \tag{22}$$

It is obvious that $\tau(0) \geq \varepsilon_2$ since $\tau(0) = \varepsilon_2 + Y_0^2$ and $\varepsilon_2 > 0$. It is easy to show that under conditions $\varepsilon_2 - \varepsilon_1 > 0$, $\varepsilon_2 - \varepsilon_3 > 0$, $a > 0$, and $b > 0$ such a root exists. Indeed, rewrite equation (22) in the following form $a_3x^3 + a_2x^2 + a_1x = a_0$, where $x = \tau(0) - \varepsilon_2$ and a_0, a_1, a_2, a_4 all are positive. Then, this equation has the root $x > 0$. This implies that equation (22) has the root $\tau(0) > \varepsilon_2$.

It should be noticed that from equation (22) we can see that if $\varepsilon_1 = \varepsilon_3$, then one of the roots of this equation is $\tau(h)$, i.e., $\tau(0) = \tau(h)$. In original variables we obtain $Y_0^2 = Y_h^2$. The situation is almost the same for the case of a linear layer (see Ch. 1). There is a slight difference here between the case of the nonlinear layer and the case of the linear layer. In the linear case it is always $Y_0^2 = Y_h^2$ when $\varepsilon_1 = \varepsilon_3$. In the nonlinear case it is only one root of equation (22).

Under our assumptions the right-hand side of the second equation of system (18) is negative. This means that function η decreases when $x \in (0, h)$. However it is possible that there are zeros of the function Y. Since Y and Z are analytical functions, so is η. This means that η has discontinuities of the second kind at the points x^*, where $Y(x^*) = 0$. These points are poles of the function η.

These poles are zeros of denominator of the first integral. Then, in these points $\tau^* = \tau(x^*)$ is such that $\eta^* = \pm\infty$.

Let us suppose that there are $(N+1)$ points of discontinuities: $x_0, ..., x_N$ on interval $x \in (0, h)$.

The properties of function $\eta = \eta(x)$ imply

$$\eta(x_i - 0) = -\infty, \quad \eta(x_i + 0) = +\infty, \quad \text{where} \quad i = \overline{0, N}. \tag{23}$$

Denote by

$$w := \left[\gamma^2 - \varepsilon_2 - a(\tau - \varepsilon_2) - b(\tau - \varepsilon_2)^2 - \eta^2\right]^{-1},$$

where $w \equiv w(\eta)$; and $\tau = \tau(\eta)$ is expressed from first integral (19).

Taking into account our hypothesis we will seek to the solutions on each interval $[0\ , x_0)$, (x_0, x_1), ..., $(x_N\ , h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0)} w d\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} w d\eta = x + c_i, & x_i \le x \le x_{i+1}, \ \ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} w d\eta = x + c_N, & x_N \le x \le h. \end{cases} \tag{24}$$

Substituting $x = 0$, $x = x_{i+1}$, and $x = x_N$ into equations (24) (into the first, the second, and the third, respectively) and taking into account (23), we find constants c_1, c_2, ..., c_{N+1}:

$$\begin{cases} c_0 = -\int\limits_{\eta(0)}^{-\infty} w d\eta; \\ c_{i+1} = \int\limits_{+\infty}^{-\infty} w d\eta - x_{i+1}, \ \ i = \overline{0, N-1}; \\ c_{N+1} = \int\limits_{+\infty}^{\eta(h)} w d\eta - h. \end{cases} \tag{25}$$

Using (25) we can rewrite equations (24) in the following form

$$\begin{cases} \int\limits_{\eta(x)}^{\eta(x_0)} w d\eta = -x + \int\limits_{\eta(0)}^{-\infty} w d\eta, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} w d\eta = x + \int\limits_{+\infty}^{-\infty} w d\eta - x_{i+1}, & x_i \le x \le x_{i+1}, \quad i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} w d\eta = x + \int\limits_{+\infty}^{\eta(h)} w d\eta - h, & x_N \le x \le h. \end{cases} \tag{26}$$

Introduce the notation $T := -\int\limits_{-\infty}^{+\infty} \omega d\eta$. It follows from formula (26) that $0 < x_{i+1} - x_i = T < h$, where $i = \overline{0, N-1}$. This implies the convergence of the improper integral (it will be proved in other way below). Now consider x in equations (26) such that all the integrals on the left side vanish (i.e. $x = x_0$, $x = x_i$, and $x = x_N$), and sum all equations (26). We obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{-\infty} w d\eta + x_0 + T - x_1 + ... + x_{N-1} + T - x_N + x_N + \int\limits_{+\infty}^{\eta(h)} w d\eta - h.$$

Finally we have

$$-\int\limits_{-\sqrt{\gamma^2-\varepsilon_3}}^{\sqrt{\gamma^2-\varepsilon_1}} w d\eta + (N+1)T = h. \tag{27}$$

Expression (27) is the DE, which holds for any finite h. Let γ be a solution of DE (27) and an eigenvalue of the problem. Then, there is an eigenfunction Y, which corresponds to the eigenvalue γ. The eigenfunction Y has $N+1$ zeros on the interval $(0, h)$.

Notice that improper integrals in DE (27) converge. Indeed, function $\tau = \tau(\eta)$ is bounded as $\eta \to \infty$ since $\tau = \varepsilon_2 + Y^2$ and Y is bounded. Then

$$|w| = \left| \frac{1}{\gamma^2 - \varepsilon_2 - a(\tau - \varepsilon_2) - b(\tau - \varepsilon_2)^2 - \eta^2} \right| \le \left| \frac{1}{\eta^2 + \alpha} \right|,$$

where $\alpha > 0$ is a constant. It is obvious that improper integral $\int\limits_{-\infty}^{+\infty} \frac{d\eta}{\eta^2+\alpha}$ converges. Converges of the improper integrals in (27) in inner points results from the requirement that right-hand side of the second equation of system (18) is negative.

Theorem 1. *The set of solutions of DE* (27) *contains the set of solutions* (*eigenvalues*) *of the boundary eigenvalue problem* (12)–(14) *with conditions* (10), (11).

Proof. It follows from the method of obtaining of DE (27) from system (18) that an eigenvalue of the problem (12)–(14) is a solution of the DE.

It is obvious that the function τ as the function with respect to η defined from first integral (19) is a multiple-valued function. This implies that not every solution of DE (27) is an eigenvalue of the problem. In other words, there can be several roots $\tau(0)$ of equation (22) such that each of them satisfies the condition $\tau(0) \geq \varepsilon_2$. Even in this case it is possible to find eigenvalues among roots of the DE. Indeed, when we find a solution γ of DE (27), we can find functions $\tau(x)$ and $\eta(x)$ from system (18) and first integral (19). From functions $\tau(x)$ and $\eta(x)$ and using formulas (16), (17) we obtain

$$Y(x) = \pm\sqrt{\tau - \varepsilon_2} \quad \text{and} \quad Z(x) = \pm\sqrt{\tau - \varepsilon_2}|\eta|. \tag{28}$$

It is an important question how to choose the sign. Let us discuss it in detail. We know that the function η monotonically decreases. If $x = x^*$ such that $\eta(x^*) = 0$ then $\eta(x^* - 0) > 0$, $\eta(x^* + 0) < 0$; and if $x = x^{**}$ such that $\eta(x^{**}) = \pm\infty$, then $\eta(x^{**} - 0) < 0$ and $\eta(x^{**} + 0) > 0$. The function η has no other points of sign's reversal. To fix the idea, assume that the initial condition is $Y_h > 0$. If $\eta > 0$, then functions Y and Z have the same signs; if $\eta < 0$, then Y and Z have different signs. Since X and Z are continuous functions we can choose necessary signs in expressions (28). Now, when we have function Y we can calculate $\tau(0) = \varepsilon_2 + Y_0^2$. If this calculated value is equal to the value calculated from equation (22), then the solution γ of the DE is an eigenvalue of the problem (and is not an eigenvalue otherwise).

If there is a unique root $\tau(0) \geq \varepsilon_2$ of equation (22), then we have the following

Theorem 2 (*of equivalence*). *If equation* (22) *has a unique solution* $\tau(0) \geq \varepsilon_2$, *then boundary eigenvalue problem* (12)–(14) *with conditions* (10), (11), (15) *has a solution* (*an eigenvalue*) *if and only if this eigenvalue is a solution of DE* (27).

The proof of this theorem results from the proof of previous theorem.

Introduce the notation $J(\gamma, k) := -\int\limits_{\eta(0)}^{\eta(h)} w d\eta + kT$, where the right-hand side is defined by DE (27) and $k = \overline{0, N+1}$.

Let

$$h_{\inf}^{k} = \inf_{\gamma^2 \in (\max(\varepsilon_1, \varepsilon_3), \varepsilon_2)} J(\gamma, k),$$

$$h_{\sup}^{k} = \sup_{\gamma^2 \in (\max(\varepsilon_1, \varepsilon_3), \varepsilon_2)} J(\gamma, k).$$

Let us formulate the sufficient condition of existence at least one eigenvalue of the problem.

Theorem 3. *Let* h *satisfies for a certain* $k = \overline{0, N+1}$ *the following two-sided inequality*

$$h_{\inf}^{k} < h < h_{\sup}^{k},$$

then the boundary eigenvalue problem (12)–(14) *with conditions* (10), (11), (15) *has at least one solution* (*an eigenvalue*).

The quantities $h_{\inf}^{k}$ and $h_{\sup}^{k}$ can be numerically calculated.

§6. Generalized Dispersion Equation

Here we derive the generalized DE which holds for any real values ε_2, a, and b. In addition the sign of the right-hand side of the second equation in system (18) (see the footnote on p. 43), and conditions $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$ or $0 < \gamma^2 < \varepsilon_2$ are not taken into account. These conditions appear in the case of a linear layer and are used for derivation of DE (27). Though in the nonlinear case it is not necessary to the limit value γ^2 from the right side. At the same time it is clear that γ^2 is limited from the left side since this limit appears from the solutions in the half-spaces (where the permittivities are constants).

Now we assume that γ satisfies one of the following inequalities

$$\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < +\infty,$$

when either ε_1 or ε_3 is positive, or

$$0 < \gamma^2 < +\infty,$$

when both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$.

At first we derive the DE from system (18) and first integral (19). After this we discuss the details of the derivation and conditions when the derivation is possible and the DE is well defined.

Using first integral (19) it is possible to integrate formally any of the equations of system (18). As earlier we integrate the second equation. We can not obtain a solution on the whole interval $(0, h)$ since the function $\eta(x)$ can have break points which belong to $(0, h)$. It is known that the function $\eta(x)$ is an analytical one. Therefore we can conclude that if $\eta(x)$ has break points when $x \in (0, h)$, then there are only break points of the second kind. It can be proved in other way. Indeed, it is easy to see that the solution of system (8) are expressed through elliptic functions. This implies that function $\eta(x)$ has finite number second-kind break points on the interval $(0, h)$ and has no other break points.

Assume that the function $\eta(x)$ on the interval $(0, h)$ has $N + 1$ break points $x_0,\ x_1,\ ...,\ x_N$.

It should be noticed that

$$\eta(x_i - 0) = \pm\infty,\ \eta(x_i + 0) = \pm\infty,$$

where $i = \overline{0,\, N}$, and signs $\pm$ in these formulas are independent and unknown.

Taking into account the above, solutions are sought on each interval $[0\ , x_0)$, (x_0, x_1), ..., $(x_N\ , h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + c_{i+1}, & x_i \le x \le x_{i+1},\ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + c_{N+1}, & x_N \le x \le h. \end{cases} \quad (29)$$

From equations (29), substituting $x = 0$, $x = x_{i+1}$, and $x = x_N$ into the first, the second, and the third equations (29), respectively, we find required constants $c_1,\ c_2,\ ...,\ c_{N+1}$:

$$\begin{cases} c_0 = -\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta; \\ c_{i+1} = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1},\ \ i = \overline{0, N-1}; \\ c_{N+1} = \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h. \end{cases} \quad (30)$$

Using (30) equations (29) take the form

$$\begin{cases} \int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = -x + \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, & x_i \le x \le x_{i+1}; \\ \int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h, & x_N \le x \le h, \end{cases} \quad (31)$$

where $i = \overline{0, N-1}$.

From formulas (31) we obtain that

$$x_{i+1} - x_i = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta, \tag{32}$$

where $i = \overline{0, N-1}$.

Expressions $0 < x_{i+1} - x_i < h < \infty$ imply that under the assumption about the break points existence the integral on the right side converges and $\int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta > 0$.

In the same way, from the first and the last equations (31) we obtain that $x_0 = \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta$ and $0 < x_0 < h$ then

$$0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta < h < \infty;$$

and $h - x_N = \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta$ and $0 < h - x_N < h$ then

$$0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta < h < \infty.$$

These considerations yield that the function $w(\eta)$ has no non-integrable singularities for $\eta \in (-\infty, \infty)$. And also this proves that the assumption about a finite number break points is true.

Now, setting $x = x_0$, $x = x_i$, and $x = x_N$ in the first, the second, and the third equations in (31), respectively, we have that al integrals on the left-sides vanish. We add all the equations in (31) to obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + x_0 + \int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} w d\eta - x_1 + ...$$

$$... + x_{N-1} + \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} w d\eta - x_N + x_N + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h. \quad (33)$$

From (33) we obtain

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta + \sum_{i=0}^{N-1} \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta = h. \quad (34)$$

It follows from formulas (32) that

$$\eta\left(x_i + 0\right) = \pm\infty \quad \text{and} \quad \eta\left(x_i - 0\right) = \mp\infty,$$

where $i = \overline{0, N}$, and it is necessary to choose the infinities of different signs.

Thus we obtain that

$$\int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} w d\eta = ... = \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} w d\eta =: T',$$

Hence $x_1 - x_0 = ... = x_N - x_{N-1}$.

Now we can rewrite equation (34) in the following form

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} f d\eta + NT' = h.$$

Let $T := -\int\limits_{-\infty}^{+\infty} w d\eta$, then we finally obtain

$$-\int\limits_{-\sqrt{\gamma^2-\varepsilon_3}}^{\sqrt{\gamma^2-\varepsilon_1}} w d\eta \pm (N+1)T = h. \quad (35)$$

Expression (35) is the DE, which holds for any finite h. Let γ be a solution of DE (35) and an eigenvalue of the problem. Then, there is an eigenfunction Y, which corresponds to the eigenvalue γ. The eigenfunction Y has $N+1$ zeros on the interval $(0,h)$. It should be noticed that for every number $N+1$ it is necessary to solve two DEs: for $N+1$ and for $-(N+1)$.

Let us formulate the following

Theorem 4. *The set of solutions of DE* (35) *contains the set of solutions* (*eigenvalues*) *of boundary eigenvalue problem* (12)–(14) *with conditions* (10), (11), (15).

The proof of this theorem is almost word-by-word coincides to the proof of the Theorem 1.

Now let us review some theoretical treatments of the derivation of DEs (27) and (35). We are going to discuss the existence and uniqueness of system's (8) solution.

Let us consider vector form (13) of system (8):

$$\mathrm{D}\mathbf{F} = \mathbf{G}(\mathbf{F}, \lambda). \tag{36}$$

Let the right-hand side $\mathbf{G}$ be defined and cotinuous in the domain $\Omega \subset \mathbb{R}^2$, $\mathbf{G} : \Omega \to \mathbb{R}^2$. Also we suppose that $\mathbf{G}$ satisfies the Lipschitz condition on $\mathbf{F}$ (locally in Ω)[1].

Under these conditions system (8) (or system (36)) has a unique solution in the domain Ω [8, 41, 22].

It is clear that under these conditions system (18) has a unique solution (of course, the domain of uniqueness Ω' for variables τ, η differs from Ω).

Since we seek bounded solutions Y and Z; therefore,

$$\Omega \subset [-m_1, m_1] \times [-m_2, m_2],$$

where

$$\max_{x\in[0,h]} |Y| < m_1, \quad \max_{x\in[0,h]} |Z| < m_2,$$

and the previous implies that

$$\Omega' \subset [\varepsilon_2, \varepsilon_2 + m_1^2] \times (-\infty, +\infty).$$

[1]About the Lipschitz condition see the footnote on p. 48

Under our assumptions the right-hand side of system (36) is analytical and, therefore, the Lipschitz condition is fulfilled. This means that for such a system all mentioned statements about existence and uniqueness of a solution hold (also see the end of the last paragraph of Ch. 3).

Note 1. If there is a certain value γ_*^2 such that some of the integrals in DEs (27) or (35) diverge in certain inner points, then this simply means that the value γ_*^2 is not a solution of chosen DE and the value γ_*^2 is not an eigenvalue of the problem.

Note 2. This problem depends on the initial condition Y_h, see the note on p. 49 for further details.

We derived the DEs from the second equation of system (18). It is possible to do it using the first equation of the system (see p. 130).

C H A P T E R 5

TM WAVE PROPAGATION IN A LINEAR LAYER

§1. Statement of the Problem

Let us consider electromagnetic waves propagating through a homogeneous isotropic nonmagnetic dielectric layer. The layer is located between two half-spaces: $x < 0$ and $x > h$ in Cartesian coordinate system $Oxyz$. The half-spaces are filled with isotropic nonmagnetic media without any sources and characterized by permittivities $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, respectively, where ε_0 is the permittivity of free space[1]. Assume that everywhere $\mu = \mu_0$, where μ_0 is the permeability of free space.

The electromagnetic field depends on time harmonically [17]

$$\tilde{\mathbf{E}}\,(x, y, z, t) = \mathbf{E}_+\,(x, y, z) \cos \omega t + \mathbf{E}_-\,(x, y, z) \sin \omega t,$$
$$\tilde{\mathbf{H}}\,(x, y, z, t) = \mathbf{H}_+\,(x, y, z) \cos \omega t + \mathbf{H}_-\,(x, y, z) \sin \omega t,$$

where ω is the circular frequency; $\tilde{\mathbf{E}}$, $\mathbf{E}_+$, $\mathbf{E}_-$, $\tilde{\mathbf{H}}$, $\mathbf{H}_+$, $\mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = (E_x, E_y, E_z)^T, \quad \mathbf{H} = (H_x, H_y, H_z)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

[1]Generally, conditions $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$ are not necessary. They are not used for derivation of DEs, but they are useful for DEs' solvability analysis.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential field components on the media interfaces $x = 0$, $x = h$ and the radiation condition at infinity: the electromagnetic field exponentially decays as $|x| \to \infty$ in the domains $x < 0$ and $x > h$.

The permittivity inside the layer is described by the diagonal tensor

$$\tilde{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix},$$

where ε_{xx}, ε_{zz} are constants. In the case of TM waves it does not matter what a form ε_{yy} has. As for TM waves the value ε_{yy} is not contained in the equations below.

The solution to the Maxwell equations are sought in the entire space.

The geometry of the problem is shown in Fig. 1.

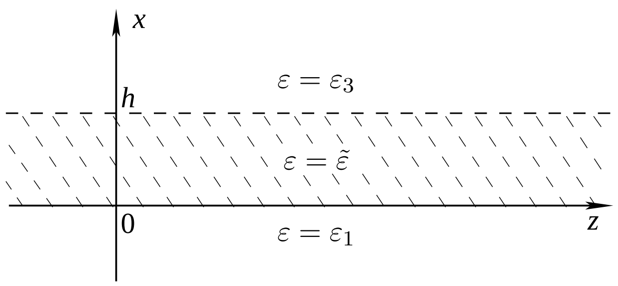


Fig. 1.

§2. TM Waves

Let us consider TM waves

$$\mathbf{E} = (E_x, 0, E_z)^T, \quad \mathbf{H} = (0, H_y, 0)^T,$$

where $E_x = E_x(x, y, z)$, $E_z = E_z(x, y, z)$, and $H_y = H_y(x, y, z)$.

Substituting the fields into Maxwell equations (1) we obtain

$$\begin{cases} \frac{\partial E_z}{\partial y} = 0, \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega\mu H_y, \\ \frac{\partial E_x}{\partial y} = 0, \\ \frac{\partial H_y}{\partial z} = i\omega\varepsilon_{xx}E_x, \\ \frac{\partial H_y}{\partial x} = -i\omega\varepsilon_{zz}E_z. \end{cases}$$

It is obvious from the first and the third equations of this system that E_z and E_x do not depend on y. This implies that H_y does not depend on y.

Waves propagating along medium interface z depend on z harmonically. This means that the fields components have the form

$$E_x = E_x(x)e^{i\gamma z}, E_z = E_z(x)e^{i\gamma z}, H_y = H_y(x)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system

$$\begin{cases} \gamma\left(iE_x(x)\right)' - E_z''(x) = \omega^2\mu\varepsilon_{zz}E_z(x), \\ \gamma^2\left(iE_x(x)\right) - \gamma E_z'(x) = \omega^2\mu\varepsilon_{xx}\left(iE_x(x)\right), \end{cases} \tag{2}$$

where $(\,\cdot\,)' \equiv \frac{d}{dx}$.

Let us denote by $k_0^2 := \omega^2\mu\varepsilon_0$, and perform the normalization according to the formulas $\tilde{x} = k_0 x$, $\frac{d}{dx} = k_0\frac{d}{d\tilde{x}}$, $\tilde{\gamma} = \frac{\gamma}{k_0}$, $\tilde{\varepsilon}_1 = \frac{\varepsilon_1}{\varepsilon_0}$, $\tilde{\varepsilon}_3 = \frac{\varepsilon_3}{\varepsilon_0}$, $\tilde{\varepsilon}_{xx} = \frac{\varepsilon_{xx}}{\varepsilon_0}$, $\tilde{\varepsilon}_{zz} = \frac{\varepsilon_{zz}}{\varepsilon_0}$. Denoting by $Z(\tilde{x}) := E_z$, $X(\tilde{x}) := iE_x$ and omitting the tilde symbol from system (2) we have

$$\begin{cases} -\frac{d^2Z}{dx^2} + \gamma\frac{dX}{dx} = \varepsilon_{zz}Z, \\ -\frac{dZ}{dx} + \gamma X = \frac{1}{\gamma}\varepsilon_{xx}X. \end{cases}$$

From this system we obtain

$$\begin{cases} X'' - \lambda X = 0, \\ Z = \frac{1}{\gamma}\frac{\varepsilon_{xx}}{\varepsilon_{zz}}X', \end{cases} \tag{3}$$

where

$$\lambda = \frac{\varepsilon_{zz}}{\varepsilon_{xx}}(\gamma^2 - \varepsilon_{xx}). \tag{4}$$

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along boundaries of the layer $0 < x < h$, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of the spectral parameter γ such that real solutions $X(x)$ and $Z(x)$ to system (3) exist.

Note. We consider that γ is a real value, but in the linear case it is possible to consider that the spectral parameter γ is a complex value. In nonlinear cases under our approach it is impossible to use complex value of γ (see the footnote on p. 33).

Also we assume that functions X and Z are sufficiently smooth

$$\begin{aligned} X(x) \in & C(-\infty,\ 0] \cap C[0,h] \cap C[h,\ +\infty) \cap \\ & \cap C^1(-\infty,\ 0] \cap C^1[0,h] \cap C^1[h,\ +\infty), \\ Z(x) \in & C(-\infty,+\infty) \cap C^1(-\infty,\ 0] \cap C^1[0,h] \cap C^1[h,\ +\infty) \cap \\ & \cap C^2(-\infty,0) \cap C^2(0,h) \cap C^2(h,+\infty). \end{aligned}$$

Physical nature of the problem implies these conditions.

System (3) is the system for the anisotropic layer. Systems for the half-spaces can be easily obtained from system (3). For this purpose in system (3) it is necessary to put $\varepsilon_{xx} = \varepsilon_{zz} = \varepsilon$, where ε is the permittivity of the isotropic half-space.

We consider that γ satisfies the inequality $\gamma^2 > \max(\varepsilon_1, \varepsilon_3)$. This condition occurs in the case if at least one of the values ε_1 or ε_3 is positive. If both values ε_1 and ε_3 are negative, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem

In the domain $x < 0$ we have $\varepsilon = \varepsilon_1$. From system (3) we obtain $X'' = \left(\gamma^2 - \varepsilon_1\right) X$. Its general solution is

$$X(x) = A_1 e^{-\sqrt{\gamma^2-\varepsilon_1}x} + A e^{\sqrt{\gamma^2-\varepsilon_1}x}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} X(x) &= Ae^{x\sqrt{\gamma^2-\varepsilon_1}}, \\ Z(x) &= \frac{\sqrt{\gamma^2-\varepsilon_1}}{\gamma}Ae^{x\sqrt{\gamma^2-\varepsilon_1}}. \end{aligned} \tag{5}$$

We assume that $\gamma^2-\varepsilon_1>0$ otherwise it will be impossible to satisfy the radiation condition.

In the domain $x>h$ we have $\varepsilon=\varepsilon_3$. From system (3) we obtain $X''=\left(\gamma^2-\varepsilon_3\right)X$. Its general solution is

$$X(x) = B_1e^{(x-h)\sqrt{\gamma^2-\varepsilon_3}} + Be^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} X(x) &= Be^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}}, \\ Z(x) &= -\frac{\sqrt{\gamma^2-\varepsilon_3}}{\gamma}Be^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}}. \end{aligned} \tag{6}$$

Here for the same reason as above we consider that $\gamma^2-\varepsilon_3>0$.

Constants A and B in (5) and (6) are defined by transmission conditions and initial conditions.

Inside the layer $0<x<h$ it is necessary to solve system (3). It is possible here to consider two cases:

a) $\lambda\geq 0$; and general solution of system (3) inside the layer is

$$\begin{aligned} X(x) &= C_1e^{-x\sqrt{\lambda}} + C_2e^{x\sqrt{\lambda}}, \\ Z(x) &= \frac{1}{\gamma}\sqrt{\frac{\varepsilon_{xx}}{\varepsilon_{zz}}(\gamma^2-\varepsilon_{xx})}\left(-C_1e^{-x\sqrt{\lambda}} + C_2e^{x\sqrt{\lambda}}\right); \end{aligned} \tag{7}$$

b) $\lambda\leq 0$; and general solution of system (3) inside the layer is

$$\begin{aligned} X(x) &= C_1\sin x\sqrt{-\lambda} + C_2\cos x\sqrt{-\lambda}, \\ Z(x) &= \frac{1}{\gamma}\sqrt{\frac{\varepsilon_{xx}}{\varepsilon_{zz}}(\varepsilon_{xx}-\gamma^2)}\left(C_1\cos x\sqrt{-\lambda} - C_2\sin x\sqrt{-\lambda}\right). \end{aligned} \tag{8}$$

§4. Transmission Conditions

Tangential components of an electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are H_y and E_z. Hence we obtain

$$\begin{aligned} &H_y\,(h+0) = H_y\,(h-0)\,, \quad H_y\,(0-0) = H_y\,(0+0)\,,\\ &E_z\,(h+0) = E_z\,(h-0)\,, \quad E_z\,(0-0) = E_z\,(0+0)\,. \end{aligned}$$

It is also known that εE_x is continuous at media interfaces, where E_x is a normal component of the electric field. This implies that εX is continuous at the media interfaces.

The continuity conditions for the tangential components of electromagnetic field, continuity condition for εX, and formulas (2), (3) imply the transmission conditions for functions X and Z

$$[\varepsilon X]_{x=0} = 0, \quad [\varepsilon X]_{x=h} = 0, \quad [Z]_{x=0} = 0, \quad [Z]_{x=h} = 0, \tag{9}$$

where $[f]_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$ denotes a jump of the function f at the interface.

Denote by $X_0 := X(0-0)$, $X_h := X(h+0)$, $Z_0 := Z(0-0)$, and $Z_h := Z(h+0)$. Then, we obtain

$$A = \frac{\gamma}{\sqrt{\gamma^2-\varepsilon_1}} Z_0, \quad B = -\frac{\gamma}{\sqrt{\gamma^2-\varepsilon_3}} Z_h$$

and also

$$X_0 = \frac{\gamma}{\sqrt{\gamma^2-\varepsilon_1}} Z_0, \quad X_h = -\frac{\gamma}{\sqrt{\gamma^2-\varepsilon_3}} Z_h.$$

The constant Z_h is supposed to be known (initial condition).

In case (a) from transmission conditions (9) and solutions (5)–(7) we obtain the system

$$\begin{cases} \varepsilon_1 A = \varepsilon_{xx}\,(C_1 + C_2)\,,\\ \varepsilon_3 B = \varepsilon_{xx}\left(C_1 e^{-h\sqrt{\lambda}} + C_2 e^{h\sqrt{\lambda}}\right),\\ \sqrt{\gamma^2-\varepsilon_1}A = \sqrt{\frac{\varepsilon_{xx}}{\varepsilon_{zz}}(\gamma^2-\varepsilon_{xx})}(C_2 - C_1),\\ -\sqrt{\gamma^2-\varepsilon_3}B = \sqrt{\frac{\varepsilon_{xx}}{\varepsilon_{zz}}(\gamma^2-\varepsilon_{xx})}\left(-C_1 e^{-h\sqrt{\lambda}} + C_2 e^{h\sqrt{\lambda}}\right). \end{cases}$$

Solving this system we obtain the DE

$$e^{2h\sqrt{\lambda}} = \frac{\varepsilon_1\sqrt{\gamma^2-\varepsilon_{xx}} - \sqrt{\varepsilon_{xx}\varepsilon_{zz}(\gamma^2-\varepsilon_1)}}{\varepsilon_1\sqrt{\gamma^2-\varepsilon_{xx}} + \sqrt{\varepsilon_{xx}\varepsilon_{zz}(\gamma^2-\varepsilon_1)}}\times \times\frac{\varepsilon_3\sqrt{\gamma^2-\varepsilon_{xx}} - \sqrt{\varepsilon_{xx}\varepsilon_{zz}(\gamma^2-\varepsilon_3)}}{\varepsilon_3\sqrt{\gamma^2-\varepsilon_{xx}} + \sqrt{\varepsilon_{xx}\varepsilon_{zz}(\gamma^2-\varepsilon_3)}}, \quad (10)$$

where $\gamma^2-\varepsilon_1>0$, $\gamma^2-\varepsilon_3>0$, and $\lambda\geq 0$.

In case (b) from transmission conditions (9) and solutions (5), (6), (8) we obtain the system

$$\begin{cases} \varepsilon_1 A = \varepsilon_{xx}C_2, \\ \varepsilon_3 B = \varepsilon_{xx}\left(C_1\sin h\sqrt{-\lambda} + C_2\cos h\sqrt{-\lambda}\right), \\ \sqrt{\gamma^2-\varepsilon_1}A = \sqrt{\frac{\varepsilon_{xx}}{\varepsilon_{zz}}(\varepsilon_{xx}-\gamma^2)}C_1, \\ -\sqrt{\gamma^2-\varepsilon_3}B = \sqrt{\frac{\varepsilon_{xx}}{\varepsilon_{zz}}(\varepsilon_{xx}-\gamma^2)}\left(C_1\cos h\sqrt{-\lambda} - C_2\sin h\sqrt{-\lambda}\right). \end{cases}$$

From this system we find

$$\frac{\varepsilon_1\varepsilon_3\left(\varepsilon_{xx}-\gamma^2\right) - \varepsilon_{xx}\varepsilon_{zz}\sqrt{\gamma^2-\varepsilon_1}\sqrt{\gamma^2-\varepsilon_3}}{\sqrt{\varepsilon_{xx}\varepsilon_{zz}(\varepsilon_{xx}-\gamma^2)}\left(\varepsilon_3\sqrt{\gamma^2-\varepsilon_1}+\varepsilon_1\sqrt{\gamma^2-\varepsilon_3}\right)}\sin h\sqrt{-\lambda} = = \cos h\sqrt{-\lambda}, \quad (11)$$

where $\gamma^2-\varepsilon_1>0$, $\gamma^2-\varepsilon_3>0$, and $\lambda\leq 0$.

If $\cos h\sqrt{-\lambda}\neq 0$, then we obtain the well-known equation

$$\operatorname{tg} h\sqrt{-\lambda} = \frac{\sqrt{\varepsilon_{xx}\varepsilon_{zz}(\varepsilon_{xx}-\gamma^2)}\left(\varepsilon_3\sqrt{\gamma^2-\varepsilon_1}+\varepsilon_1\sqrt{\gamma^2-\varepsilon_3}\right)}{\varepsilon_1\varepsilon_3\left(\varepsilon_{xx}-\gamma^2\right) - \varepsilon_{xx}\varepsilon_{zz}\sqrt{\gamma^2-\varepsilon_1}\sqrt{\gamma^2-\varepsilon_3}}. \quad (12)$$

If the condition $\cos h\sqrt{-\lambda}\neq 0$ does not hold, then we can find the simpler (algebraic) equation for γ.

Equation (10) can be formally derived from (11). Indeed, if we simply change in (11) $-\lambda$ by λ, take into account occurred imaginary unit, then we obtain (10). In the same way it is possible to derive (11) from (10).

§5. Analysis of Dispersion Equations

In both DE (10) and (12) from the conditions $\gamma^2 - \varepsilon_1 > 0$ and $\gamma^2 - \varepsilon_3 > 0$ follow that ε_1 and ε_3 can have arbitrary signs (this is mentioned in §1).

Let us consider the routine case when $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, where ε_0 is the permittivity of free space.

The conditions $\gamma^2 - \varepsilon_1 > 0$ and $\gamma^2 - \varepsilon_3 > 0$ imply that

$$\gamma^2 > \max(\varepsilon_1, \varepsilon_3).$$

The condition $\lambda > 0$ implies the following inequalities

$$\begin{cases} \varepsilon_{xx} > 0, \\ \varepsilon_{zz} > 0, \\ \gamma^2 > \varepsilon_{xx}, \end{cases} \quad \text{or} \quad \begin{cases} \varepsilon_{xx} < 0, \\ \varepsilon_{zz} < 0, \\ \gamma^2 > \varepsilon_{xx}, \end{cases} \quad \text{or} \quad \begin{cases} \varepsilon_{xx} > 0, \\ \varepsilon_{zz} < 0, \\ \gamma^2 < \varepsilon_{xx}. \end{cases} \tag{13}$$

The condition $\lambda < 0$ implies the following inequalities

$$\begin{cases} \varepsilon_{xx} > 0, \\ \varepsilon_{zz} < 0, \\ \gamma^2 > \varepsilon_{xx}, \end{cases} \quad \text{or} \quad \begin{cases} \varepsilon_{xx} < 0, \\ \varepsilon_{zz} > 0, \\ \gamma^2 > \varepsilon_{xx}, \end{cases} \quad \text{or} \quad \begin{cases} \varepsilon_{xx} > 0, \\ \varepsilon_{zz} > 0, \\ \gamma^2 < \varepsilon_{xx}. \end{cases} \tag{14}$$

It should be noticed that the first or the second group of inequalities in (14) implies that $\gamma^2 > \max(\varepsilon_{xx}, \varepsilon_1, \varepsilon_3)$. This means that the value γ^2 is not a bounded quantity. It can be proved that in equation (12) $\lim\limits_{\gamma^2 \to +\infty} h = 0$ (Fig. 2).

Also the first or the second group of inequalities in (13) implies that $\gamma^2 > \max(\varepsilon_{xx}, \varepsilon_1, \varepsilon_3)$. This means that the value γ^2 is not a bounded quantity. It is necessary to keep in mind that in the case of equation (10) only one dispersion curve exists, as exponential function has an imaginary period. This is the difference between the TM waves case and analogous case for TE waves. It is proved in Ch. 2 that for TE waves either $0 < \gamma^2 < \varepsilon_2$ or $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$ is fulfilled.

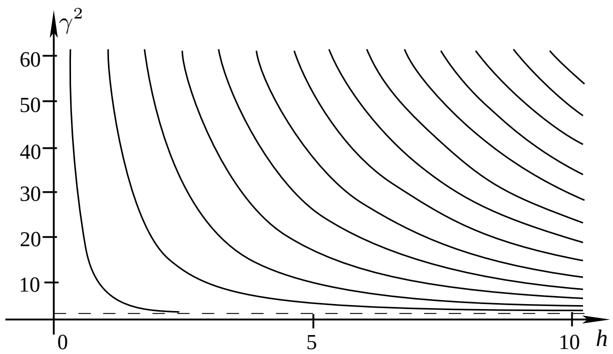


Fig. 2. $\varepsilon_1 = 1$, $\varepsilon_{xx} = 3$, $\varepsilon_{zz} = -1$, $\varepsilon_3 = 2$.
The dashed line corresponds to $\gamma^2 = \varepsilon_{xx}$

Let $\varepsilon_{xx} = \varepsilon_{zz} = \varepsilon_2$ and analyze equations (10), (12).
From equation (10) we obtain

$$h = \frac{\ln\left(\frac{\varepsilon_1\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_1}}{\varepsilon_1\sqrt{\gamma^2-\varepsilon_2}+\varepsilon_2\sqrt{\gamma^2-\varepsilon_1}} \cdot \frac{\varepsilon_3\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_3}}{\varepsilon_3\sqrt{\gamma^2-\varepsilon_2}+\varepsilon_2\sqrt{\gamma^2-\varepsilon_3}}\right)}{2\sqrt{\gamma^2-\varepsilon_2}} + \frac{i\pi k}{\sqrt{\gamma^2-\varepsilon_2}}, \quad (15)$$

where $k \in \mathbb{Z}$ and $\gamma^2 - \varepsilon_1 > 0$, $\gamma^2 - \varepsilon_2 > 0$, $\gamma^2 - \varepsilon_3 > 0$.

Inequalities (13) take the form

$$\begin{cases} \varepsilon_2 > 0, \\ \gamma^2 > \varepsilon_2, \end{cases} \quad \text{or} \quad \begin{cases} \varepsilon_2 < 0, \\ \gamma^2 > \varepsilon_2. \end{cases} \quad (16)$$

Inequalities (16) imply that in (15) $k = 0$.

It is easy to see from equation (15) that if $\varepsilon_1 < 0$, $\varepsilon_2 < 0$, and $\varepsilon_3 < 0$, then modulus of the value under the logarithm sign is less than 1. This means that the value h has negative or imaginary value. It is also true under conditions $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and $\varepsilon_3 > 0$. In other cases the value h can have either negative or positive value. The value h is the thickness of the layer and hence $h > 0$.

We are interested in the case when $\varepsilon_1 \geq \varepsilon_0$, $\varepsilon_3 \geq \varepsilon_0$, where ε_0 is the permittivity of free space and $\varepsilon_2 < 0$. This implies that

$\gamma^2 > \max(\varepsilon_1, \varepsilon_3)$. Let us consider equation (10) under these conditions. Immediately notice that in this case $\left|\frac{\varepsilon_1\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_1}}{\varepsilon_1\sqrt{\gamma^2-\varepsilon_2}+\varepsilon_2\sqrt{\gamma^2-\varepsilon_1}}\right| > 1$ and $\left|\frac{\varepsilon_3\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_3}}{\varepsilon_3\sqrt{\gamma^2-\varepsilon_2}+\varepsilon_2\sqrt{\gamma^2-\varepsilon_3}}\right| > 1$. This means that modulus of the value under the logarithm sign in (15) is more than 1. And now it is necessary to find the conditions when the value under logarithm sign is positive. Formula (15) can be rewritten in the following form

$$h = \frac{\ln\left(\frac{\left(\varepsilon_1\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_1}\right)^2}{\varepsilon_1^2(\gamma^2-\varepsilon_2)-\varepsilon_2^2(\gamma^2-\varepsilon_1)}\cdot\frac{\left(\varepsilon_3\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_3}\right)^2}{\varepsilon_3^2(\gamma^2-\varepsilon_2)-\varepsilon_2^2(\gamma^2-\varepsilon_3)}\right)}{2\sqrt{\gamma^2-\varepsilon_2}}. \tag{17}$$

It is obvious from formula (17) that the denominators must have the same signs. Rewrite formula (17) in the form

$$h = \frac{\ln\frac{\left(\varepsilon_1\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_1}\right)^2\left(\varepsilon_3\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_3}\right)^2}{(\varepsilon_1-\varepsilon_2)(\varepsilon_3-\varepsilon_2)(\gamma^2(\varepsilon_1+\varepsilon_2)-\varepsilon_1\varepsilon_2)(\gamma^2(\varepsilon_3+\varepsilon_2)-\varepsilon_3\varepsilon_2)}}{2\sqrt{\gamma^2-\varepsilon_2}}. \tag{18}$$

It is obvious that $\varepsilon_1-\varepsilon_2 > 0$ and $\varepsilon_3-\varepsilon_2 > 0$. This means that we only have to study $\left(\gamma^2(\varepsilon_1+\varepsilon_2)-\varepsilon_1\varepsilon_2\right)$ and $\left(\gamma^2(\varepsilon_3+\varepsilon_2)-\varepsilon_3\varepsilon_2\right)$. These multipliers must be either simultaneously negative or positive. If both multipliers are negative, then it is easy to show that $\gamma^2 > \max\left(\varepsilon_1, \varepsilon_3, \frac{\varepsilon_1|\varepsilon_2|}{|\varepsilon_2|-\varepsilon_1}, \frac{\varepsilon_3|\varepsilon_2|}{|\varepsilon_2|-\varepsilon_3}\right)$. Since $\frac{|\varepsilon_2|}{|\varepsilon_2|-\varepsilon_1} > 1$ and $\frac{|\varepsilon_2|}{|\varepsilon_2|-\varepsilon_3} > 1$; therefore, we obtain finally

$$\gamma^2 > \gamma_*^2 = |\varepsilon_2|\cdot\max\left(\frac{\varepsilon_1}{|\varepsilon_2|-\varepsilon_1}, \frac{\varepsilon_3}{|\varepsilon_2|-\varepsilon_3}\right).$$

It can be proved that in this case $\lim\limits_{\gamma^2\to\infty} h = 0$. Also it is easy to see that $\lim\limits_{\gamma^2\to\gamma_*^2+0} h = +\infty$ (Fig. 3).

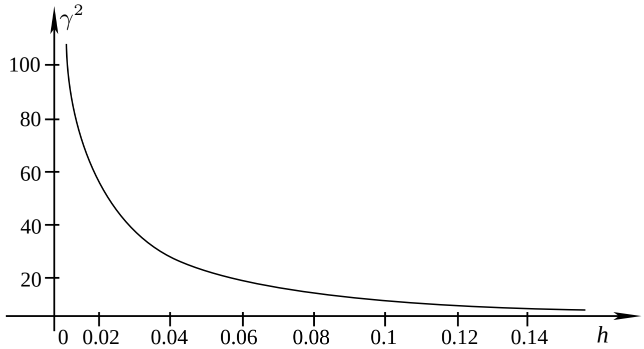


Fig. 3. $\varepsilon_1 = 3$, $\varepsilon_2 = -5$, $\varepsilon_3 = 2$

If both multipliers are positive, then there are 4 cases:

a) $|\varepsilon_2| < \min(\varepsilon_1, \varepsilon_3)$ and $\gamma^2 > \max(\varepsilon_1, \varepsilon_3)$. In this case we obtain $\lim\limits_{\gamma^2 \to \infty} h = 0$ (since the value under logarithm sign in (15) is more than 1 and bounded; and the multiplier before logarithm sign tends to 0);

b) $\varepsilon_3 < |\varepsilon_2| < \varepsilon_1$ and $\max(\varepsilon_1, \varepsilon_3) = \varepsilon_1 < \gamma^2 < \frac{\varepsilon_3|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_3}$. Then we obtain $\varepsilon_1 < \frac{\varepsilon_3|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_3}$. Under $\gamma^2 > \frac{\varepsilon_3|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_3}$ we obtain imaginary value for h. It is obvious from formula (18) that $\lim\limits_{\gamma^2 \to \frac{\varepsilon_3|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_3} - 0} h = +\infty$, i.e., there is the asymptote $\gamma^2 = \frac{\varepsilon_3|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_3}$. The value h under $\gamma^2 \to \varepsilon_1 + 0$ has finite value (Fig. 4,b)[1];

c) $\varepsilon_1 < |\varepsilon_2| < \varepsilon_3$ and $\max(\varepsilon_1, \varepsilon_3) = \varepsilon_3 < \gamma^2 < \frac{\varepsilon_1|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_1}$. Then we obtain $\varepsilon_3 < \frac{\varepsilon_1|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_1}$. Under $\gamma^2 > \frac{\varepsilon_1|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_1}$ the value h has imaginary value;

d) $|\varepsilon_2| > \max(\varepsilon_1, \varepsilon_3)$, $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \min\left(\frac{\varepsilon_1|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_1}, \frac{\varepsilon_3|\varepsilon_2|}{|\varepsilon_2| - \varepsilon_3}\right)$.

It should be noticed that the two-sided inequality in the case d for certain values of the parameters can be contradictory. For example, under $\varepsilon_2 = -5$, $\varepsilon_1 = \varepsilon_3 = 1$ we obtain $2 < \gamma^2 < 5/4$.

Each of the cases is shown in corresponding Fig. 4,a–d.

[1] The conclusion about the asymptote existence is true for the cases c, d.

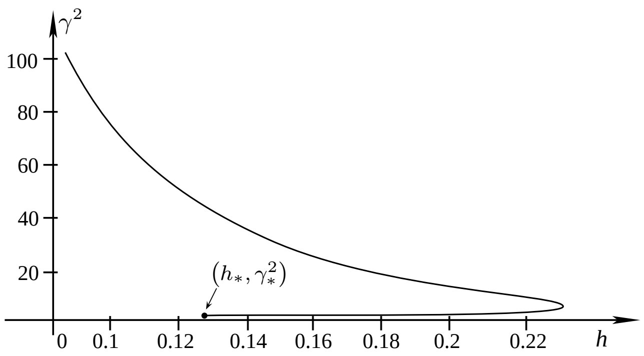


Fig. 4,a. $\varepsilon_1 = 3$, $\varepsilon_2 = -1$, $\varepsilon_3 = 2$; $\gamma_*^2 = 3$, $h_* = h\left(\gamma_*^2\right) = \frac{1}{4} \ln \frac{5}{3}$

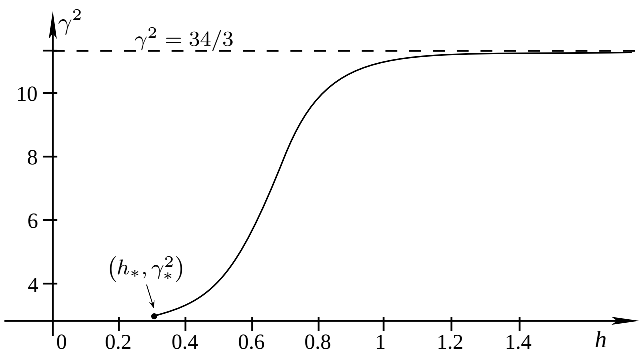


Fig. 4,b. $\varepsilon_1 = 3$, $\varepsilon_2 = -2$, $\varepsilon_3 = 2$; $\gamma_*^2 = 3$, $h_* = h\left(\gamma_*^2\right) = \frac{1}{2\sqrt{5}} \ln \frac{393+68\sqrt{26}}{185}$

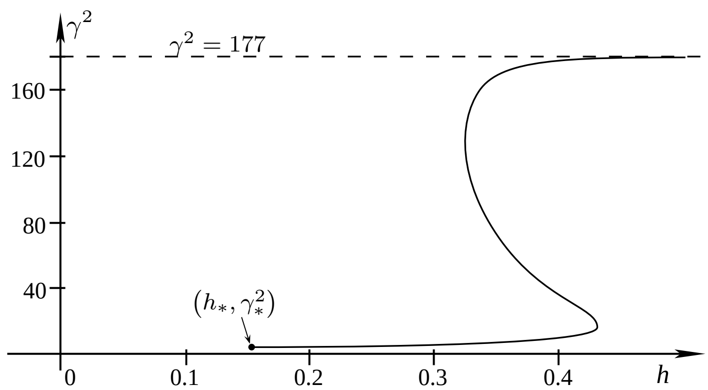


Fig. 4,c. $\varepsilon_1 = 2.95$, $\varepsilon_2 = -3$, $\varepsilon_3 = 4$; $\gamma_*^2 = 4$, $h_* = h\left(\gamma_*^2\right) \approx 0.157370723$

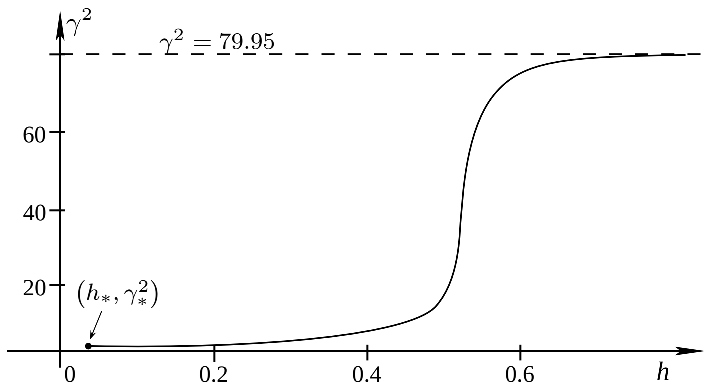


Fig. 4,d. $\varepsilon_1 = 3.9$, $\varepsilon_2 = -4.1$, $\varepsilon_3 = 4$; $\gamma_*^2 = 4$, $h_* = h\left(\gamma_*^2\right) \approx 0.041230750$

As it is clear from the computation above cases b–d do not appreciably distinguish. This fact is shown in Fig. 4,b–d. The curves in these cases are very similar to each other. And it is possible to make them practically identical if the parameters ε_1, ε_2, and ε_3 are specially chosen.

In regard to determination of eigenvalues using figures see p. 29.

Now let us go over to equation (12). This is classical equation and under $\varepsilon_{xx} = \varepsilon_{zz} = \varepsilon_2$, $\varepsilon_1 = \varepsilon_3$ is cited in [64]. Under conditions $\varepsilon_{xx} = \varepsilon_{zz} = \varepsilon_2$ equation (12) takes the form

$$\operatorname{tg} h\sqrt{\varepsilon_2 - \gamma^2} = \frac{\varepsilon_2\sqrt{\varepsilon_2 - \gamma^2}\left(\varepsilon_3\sqrt{\gamma^2 - \varepsilon_1} + \varepsilon_1\sqrt{\gamma^2 - \varepsilon_3}\right)}{\varepsilon_1\varepsilon_3\left(\varepsilon_2 - \gamma^2\right) - \varepsilon_2^2\sqrt{\gamma^2 - \varepsilon_1}\sqrt{\gamma^2 - \varepsilon_3}}. \tag{19}$$

Inequalities (14) take the form

$$\begin{cases} \varepsilon_2 > 0, \\ \gamma^2 < \varepsilon_2. \end{cases} \tag{20}$$

Equation (19) and inequalities (20) imply that

$$\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2. \tag{21}$$

Introduce the notation $\varepsilon^* := \max(\varepsilon_1, \varepsilon_3)$, $\varepsilon_* := \min(\varepsilon_1, \varepsilon_3)$ and hence $h_* := \lim\limits_{\gamma^2 \to \varepsilon^*} h(\gamma) = \frac{1}{\sqrt{\varepsilon_2 - \varepsilon^*}} \operatorname{arctg} \frac{\varepsilon_2\sqrt{\varepsilon^* - \varepsilon_*}}{\varepsilon_*\sqrt{\varepsilon_2 - \varepsilon^*}}$. It is obvious that $0 < h_* < +\infty$. The less is the value $\varepsilon_2 - \varepsilon^*$ the more is the value h_*.

Conclusion. There are only finite number of waves propagating through a layer with constant permittivity (this number is equal to a number of eigenvalues). The more is the value h the more waves propagate in this layer. If $\varepsilon_* \neq \varepsilon^*$ (in other words, if $\varepsilon_1 \neq \varepsilon_3$), then there is $h_* > 0$ such that there are no waves in the layer with $h < h_*$. This conclusion holds only for a linear waveguide structure (all these hold for equation (19)).

This conclusion holds only for a linear waveguide structure. In the case of a nonlinear layer it is possible that for any value h infinite number of propagating waves exist, i.e., there are infinite number of eigenvalues.

In the case of equation (19) the behavior of the DCs is the same as it is shown in Fig. 1, 2 in Ch. 2 (p. 29, 29).

CHAPTER 6

TM WAVE PROPAGATION IN A LAYER WITH ARBITRARY NONLINEARITY

§1. Statement of the Problem

Let us consider electromagnetic waves propagating through a homogeneous isotropic nonmagnetic dielectric layer. The layer is located between two half-spaces: $x < 0$ and $x > h$ in Cartesian coordinate system $Oxyz$. The half-spaces are filled with isotropic nonmagnetic media without any sources and characterized by permittivities $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, respectively, where ε_0 is the permittivity of free space[1]. Assume that everywhere $\mu = \mu_0$, where μ_0 is the permeability of free space.

The electromagnetic field depends on time harmonically [17]

$$\tilde{\mathbf{E}}(x,y,z,t) = \mathbf{E}_+(x,y,z)\cos\omega t + \mathbf{E}_-(x,y,z)\sin\omega t,$$
$$\tilde{\mathbf{H}}(x,y,z,t) = \mathbf{H}_+(x,y,z)\cos\omega t + \mathbf{H}_-(x,y,z)\sin\omega t,$$

where ω is the circular frequency; $\mathbf{E}$, $\mathbf{E}_+$, $\mathbf{E}_-$, $\mathbf{H}$, $\mathbf{H}_+$, $\mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = (E_x, E_y, E_z)^T, \quad \mathbf{H} = (H_x, H_y, H_z)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

[1]Generally, conditions $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$ are not necessary. They are not used for derivation of DEs, but they are useful for DEs' solvability analysis.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\tilde{\varepsilon}\mathbf{E}; \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential field components on the media interfaces $x = 0$, $x = h$ and the radiation condition at infinity: the electromagnetic field exponentially decays as $|x| \to \infty$ in the domains $x < 0$ and $x > h$.

The permittivity inside the layer is described by the diagonal tensor

$$\tilde{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix},$$

where

$$\varepsilon_{xx} = \varepsilon_f + \varepsilon_0 f\left(|E_x|^2, |E_z|^2\right), \ \varepsilon_{zz} = \varepsilon_g + \varepsilon_0 g\left(|E_x|^2, |E_z|^2\right).$$

In the case of TM waves it does not matter what a form ε_{yy} has. As for TM waves the value ε_{yy} are not contained in the equations below.

It is assumed that $\varepsilon_f > \max(\varepsilon_1, \varepsilon_3)$, $\varepsilon_g > \max(\varepsilon_1, \varepsilon_3)$ are constants parts of the permittivity $\tilde{\varepsilon}$. The functions f, g are analytical ones[1] and such that the relation $\frac{\partial f}{\partial\left(|E_z|^2\right)} = \frac{\partial g}{\partial\left(|E_x|^2\right)}$ is satisfied (this relation yields the total differential equation).

The relation $\frac{\partial f}{\partial\left(|E_z|^2\right)} = \frac{\partial g}{\partial\left(|E_x|^2\right)}$ for components of tensor $\tilde{\varepsilon}$ has been pointed out in [23]. Authors in [23] stated that many types of nonlinearities satisfy this relation. Using an integrating multiplier this relation can be generalized (this is also mentioned in [23]).

Also till §6 we assume that $\varepsilon_{xx} > 0$.

The solutions to the Maxwell equations are sought in the entire space.

The geometry of the problem is shown in Fig. 1.

[1]Everywhere below when we consider an analytical function we mean that it is an analytical function of a real variable.

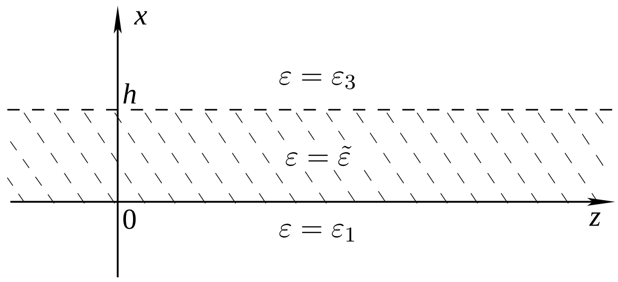


Fig. 1.

§2. TM Waves

Let us consider TM waves

$$\mathbf{E} = (E_x, 0, E_z)^T, \quad \mathbf{H} = (0, H_y, 0)^T,$$

where $E_x = E_x(x, y, z)$, $E_z = E_z(x, y, z)$, and $H_y = H_y(x, y, z)$.

Substituting the fields into Maxwell equations (1) we obtain

$$\begin{cases} \frac{\partial E_z}{\partial y} = 0, \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega\mu H_y, \\ \frac{\partial E_x}{\partial y} = 0, \\ \frac{\partial H_y}{\partial z} = i\omega\varepsilon_{xx} E_x, \\ \frac{\partial H_y}{\partial x} = -i\omega\varepsilon_{zz} E_z. \end{cases}$$

It is obvious from the first and the third equations of this system that E_z and E_x do not depend on y. This implies that H_y does not depend on y.

Waves propagating along medium interface z depend on z harmonically. This means that the fields components have the form

$$E_x = E_x(x)e^{i\gamma z}, E_z = E_z(x)e^{i\gamma z}, H_y = H_y(x)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system

$$\begin{cases} i\gamma E_x - E_z' = i\omega\mu H_y, \\ i\gamma H_y = i\omega\varepsilon_{xx} E_x, \\ H_y' = -i\omega\varepsilon_{zz} E_z. \end{cases} \tag{2}$$

The following system can be easily derived from the previous system [17]

$$\begin{cases} \gamma \left(iE_x(x)\right)' - E_z''(x) = \omega^2\mu\varepsilon_{zz} E_z(x), \\ \gamma^2 \left(iE_x(x)\right) - \gamma E_z'(x) = \omega^2\mu\varepsilon_{xx} \left(iE_x(x)\right), \end{cases} \tag{3}$$

where $(\,\cdot\,)' \equiv \frac{d}{dx}$.

Let us denote by $k_0^2 := \omega^2\mu_0\varepsilon_0$ and perform the normalization according to the formulas $\tilde{x} = k_0 x$, $\frac{d}{dx} = k_0 \frac{d}{d\tilde{x}}$, $\tilde{\gamma} = \frac{\gamma}{k_0}$, $\tilde{\varepsilon}_i = \frac{\varepsilon_i}{\varepsilon_0}$ $(i = 1, 2)$, $\tilde{\varepsilon}_f = \frac{\varepsilon_f}{\varepsilon_0}$, $\tilde{\varepsilon}_g = \frac{\varepsilon_g}{\varepsilon_0}$. Denoting by $Z(\tilde{x}) := E_z$, $X(\tilde{x}) := iE_x$ and omitting the tilde symbol, we obtain from system (3)

$$\begin{cases} -\frac{d^2 Z}{dx^2} + \gamma \frac{dX}{dx} = \varepsilon_{zz} Z, \\ -\frac{dZ}{dx} + \gamma X = \frac{1}{\gamma}\varepsilon_{xx} X. \end{cases} \tag{4}$$

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along boundaries of the layer $0 < x < h$, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of spectral parameter γ such that real solutions $X(x)$ and $Z(x)$ to system (4) exist (see the footnote on p. 33 and the note on p. 74). Also we assume that $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_f$. This two-sided inequality naturally appears for an analogous problem in a layer with constant permittivity tensor (for details see Ch. 5, formula (14)).

Also we assume that functions X and Z are sufficiently smooth

$$\begin{aligned} X(x) \in & C\,(-\infty,\ 0] \cap C[0,h] \cap C\,[h,\ +\infty) \cap \\ & \cap C^1\,(-\infty,\ 0] \cap C^1[0,h] \cap C^1\,[h,\ +\infty)\,; \\ Z(x) \in & C(-\infty,+\infty) \cap C^1\,(-\infty,\ 0] \cap C^1[0,h] \cap C^1\,[h,\ +\infty) \cap \\ & \cap C^2(-\infty,0) \cap C^2(0,h) \cap C^2(h,+\infty). \end{aligned}$$

Physical nature of the problem implies these conditions.

It is clear that system (4) is an autonomous one. System (4) can be rewritten in a normal form (it will be done below). This system in the normal form can be considered as a dynamical system with analytical with respect to X and Z right-hand sides[1]. It is well known (see, for example [5]) that the solution X and Z of such a system are analytical functions with respect to independent variable as well. For this very reason we require that functions f and g are analytical. This is an important fact for DEs' derivation.

System (4) is the system for the anisotropic layer. Systems for the half-spaces can be easily obtained from system (4). For this purpose in system (4) it is necessary to put $\varepsilon_{xx} = \varepsilon_{zz} = \varepsilon$, where ε is the permittivity of the isotropic half-space.

We consider that γ satisfies the inequality $\gamma^2 > \max(\varepsilon_1, \varepsilon_3)$. This condition occurs in the case if at least one of the values ε_1 or ε_3 is positive. If both values ε_1 and ε_3 are negative, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem

In the half-spaces $x < 0$ and $x > h$ the permittivity $\tilde{\varepsilon}$ is a constant: ε_1 for $x < 0$ and ε_3 for $x > h$. Taking it into account for system (4). In both cases we obtain systems of linear differential equations.

In the domain $x < 0$ we have $\varepsilon = \varepsilon_1$. From system (4) we obtain the following system $X' = \gamma Z$, $Z' = \frac{\gamma^2 - \varepsilon_1}{\gamma} X$. From this system we obtain the equation $X'' = (\gamma^2 - \varepsilon_1)X$. Its general solution is $X(x) = A_1 e^{-x\sqrt{\gamma^2-\varepsilon_1}} + A e^{x\sqrt{\gamma^2-\varepsilon_1}}$. In accordance with the radiation condition we obtain the solution of the system

$$\begin{aligned} X(x) &= A \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right), \\ Z(x) &= \gamma^{-1}\sqrt{\gamma^2 - \varepsilon_1} A \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right). \end{aligned} \tag{5}$$

[1]Of course in the domain where these right-hand sides are analytical with respect to X and Z.

We assume that $\gamma^2 - \varepsilon_1 > 0$ otherwise it will be impossible to satisfy the radiation condition.

In the domain $x > h$ we have $\varepsilon = \varepsilon_3$. From system (4) we obtain the following system $X' = \gamma Z$, $Z' = \frac{\gamma^2-\varepsilon_3}{\gamma}X$. From this system we obtain the equation $X'' = (\gamma^2 - \varepsilon_3)X$. Its general solution is $X(x) = Be^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}} + B_1 e^{(x-h)\sqrt{\gamma^2-\varepsilon_3}}$. In accordance with the radiation condition we obtain the solution of the system

$$\begin{aligned} &X(x) = B\exp\left(-(x-h)\sqrt{\gamma^2-\varepsilon_3}\right), \\ &Z(x) = -\gamma^{-1}\sqrt{\gamma^2-\varepsilon_3}B\exp\left(-(x-h)\sqrt{\gamma^2-\varepsilon_3}\right). \end{aligned} \tag{6}$$

Here for the same reason as above we consider that $\gamma^2 - \varepsilon_3 > 0$.

Constants A and B in (5) and (6) are defined by transmission conditions and initial conditions.

Inside the layer $0 < x < h$ system (4) takes the form

$$\begin{cases} -\frac{d^2 Z}{dx^2} + \gamma\frac{dX}{dx} = (\varepsilon_g + g)\, Z, \\ -\frac{dZ}{dx} + \gamma X = \frac{1}{\gamma}(\varepsilon_f + f)\, X, \end{cases} \tag{7}$$

further the arguments of the functions f and g will be often omitted (if there is no misunderstanding).

Differentiating the second equation of this system with respect to x, we obtain

$$-Z'' + \gamma X' = \gamma^{-1}\left(2XX'f'_u + 2ZZ'f'_v\right)X + \gamma^{-1}(\varepsilon_f + f)\,X',$$

where $f'_u := f'_{X^2}$, $f'_v := f'_{Z^2}$ (further these derivatives are understood in this sense, while other sense will not be pointed out).

Using the latter equation system (7) can be rewritten in the form[1]

$$\begin{cases} \frac{dX}{dx} = \frac{\gamma^2(\varepsilon_g+g)+2\left(\varepsilon_f-\gamma^2+f\right)X^2 f'_v}{\gamma\left(2X^2 f'_u+\varepsilon_f+f\right)}Z, \\ \frac{dZ}{dx} = \frac{1}{\gamma}\left(\gamma^2 - \varepsilon_f - f\right)X. \end{cases} \tag{8}$$

[1]Now system (7) is written in a normal form. About analyticity of the solutions of this very system is said in the end of §2.

Dividing the first equation in system (8) to the second one we obtain the ordinary differential equation

$$\gamma\left(2X^2f'_u+\varepsilon_f+f\right)\frac{dX}{dZ}= \\ =\frac{\gamma^2\left(\varepsilon_g+g\right)Z+2\left(\varepsilon_f-\gamma^2+f\right)X^2Zf'_v}{\frac{1}{\gamma}\left(\gamma^2-\varepsilon_f-f\right)X}. \quad (9)$$

Equation (9) can be transformed into a total differential equation. Indeed, rewrite it into a symmetric form

$$MdX+NdZ=0,$$

where

$$M=\left(\gamma^2-\varepsilon_f-f\right)\left(2X^2f'_u+\varepsilon_f+f\right)X,$$
$$N=\left(2\left(\gamma^2-\varepsilon_f-f\right)X^2f'_v-\gamma^2\left(\varepsilon_g+g\right)\right)Z.$$

It is easy to check that the relation $\frac{\partial M}{\partial Z}=\frac{\partial N}{\partial X}$ is satisfied. This means that equation (9) can be rewritten as a total differential equation (the equation $MdX+NdZ=0$ is the total differential equation). Let us find its solution $U(X,Z)$ (it is the first integral of system (9)). Since $\frac{\partial U}{\partial x}=M$; therefore,

$$U=\int\left(\gamma^2-\varepsilon_f-f\right)\left(2X^2f'_u+\varepsilon_f+f\right)XdX+\varphi(Z)= \\ =\int X^2\left(\gamma^2-\varepsilon_f-f\right)f'_u2XdX+ \\ +\int\left(\gamma^2-\varepsilon_f-f\right)\left(\varepsilon_f+f\right)XdX+\varphi(Z).$$

Using partial integration for the first term we obtain

$$U=-\frac{1}{2}X^2\left(\gamma^2-\varepsilon_f-f\right)^2+\int X\left(\gamma^2-\varepsilon_f-f\right)^2dX+ \\ +\int\left(\gamma^2-\varepsilon_f-f\right)\left(\varepsilon_f+f\right)XdX+\varphi(Z)=$$

$$= -\frac{1}{2}X^2\left(\gamma^2 - \varepsilon_f - f\right)^2 + \int X\left(\gamma^2 - \varepsilon_f - f\right)^2 dX +$$
$$+ \int \left(\gamma^2 - \varepsilon_f - f\right)\left(-\gamma^2 + \varepsilon_f + f + \gamma^2\right) XdX + \varphi(Z) =$$
$$= -\frac{1}{2}X^2\left(\gamma^2 - \varepsilon_f - f\right)^2 + \gamma^2 \int \left(\gamma^2 - \varepsilon_f - f\right) XdX + \varphi(Z).$$

Taking into account that $\frac{\partial U}{\partial Z} = N$ we obtain

$$\frac{\partial U}{\partial Z} = 2X^2 Z\left(\gamma^2 - \varepsilon_f - f\right) f'_v - 2\gamma^2 \int XZf'_v dX + \varphi'(Z) =$$
$$= \left(2\left(\gamma^2 - \varepsilon_f - f\right) X^2 f'_v - \gamma^2\left(\varepsilon_g + g\right)\right) Z.$$

It follows from the above that

$$\varphi'(Z) = 2\gamma^2 \int XZf'_v dX - \gamma^2\left(\varepsilon_g + g\right) Z.$$

Integrating with respect to Z we obtain

$$\varphi(Z) = 2\gamma^2 \int\int XZf'_v dXdZ - \gamma^2 \int (\varepsilon_g + g)ZdZ.$$

Changing the order of integration in the double integral (Fubini theorem) we obtain

$$\varphi(Z) = \gamma^2 \int X\left(\int 2Zf'_v dZ\right) dX - \gamma^2 \int \left(\varepsilon_g + g\right) ZdZ =$$
$$= \gamma^2 \int XfdX - \gamma^2 \int \left(\varepsilon_g + g\right) ZdZ.$$

We obtain U in the following form

$$U = -\frac{1}{2}X^2\left(\gamma^2 - \varepsilon_f - f\right)^2 + \gamma^2 \int \left(\gamma^2 - \varepsilon_f - f\right) XdX +$$
$$+ \gamma^2 \int XfdX - \gamma^2 \int \left(\varepsilon_g + g\right) ZdZ.$$

From the above formula we obtain

$$U = -\frac{1}{2}X^2\left(\gamma^2 - \varepsilon_f - f\right)^2 + \\ + \frac{\gamma^2}{2}\left(\left(\gamma^2 - \varepsilon_f\right)X^2 - \varepsilon_g Z^2\right) - \gamma^2 \int ZgdZ.$$

Finally we obtain

$$U = X^2\left(\varepsilon_f - \gamma^2 + f\right)^2 + \\ + \gamma^2\left(\left(\varepsilon_f - \gamma^2\right)X^2 + \varepsilon_g Z^2\right) + \gamma^2 \int g\left(X^2, s\right)ds\bigg|_{s=Z^2} \equiv C.$$

The function $U(X, Z)$ is the first integral of system (8). We are going to use the first integral in the following form

$$X^2\left(\varepsilon_f - \gamma^2 + f\right)^2 + \gamma^2\left(\left(\varepsilon_f - \gamma^2\right)X^2 + \varepsilon_g Z^2\right) + \gamma^2 G \equiv C, \quad (10)$$

where $G = G\left(X^2, Z^2\right) \equiv \int g\left(X^2, s\right)ds\big|_{s=Z^2}$ and C is a constant of integration.

§4. Transmission Conditions and the Transmission Problem

Tangential components of an electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are H_y and E_z. Hence we obtain

$$\begin{array}{ll} H_y(h+0) = H_y(h-0), & H_y(0-0) = H_y(0+0), \\ E_z(h+0) = E_z(h-0), & E_z(0-0) = E_z(0+0). \end{array}$$

From the continuity conditions for the tangential components of the fields we obtain

$$\begin{array}{ll} \gamma X(h) - Z'(h) = H_y^{(h)}, & \gamma X(0) - Z'(0) = H_y^{(0)}, \\ Z(h) = E_z(h+0) = E_z^{(h)}, & Z(0) = E_z(0-0) = E_z^{(0)}, \end{array} \quad (11)$$

where $H_y^{(h)} := i\frac{\sqrt{\mu}}{\sqrt{\varepsilon_0}}H_y(h+0)$, $H_y^{(0)} := i\frac{\sqrt{\mu}}{\sqrt{\varepsilon_0}}H_y(0-0)$.

The constant $E_z^{(h)} = Z(h) = Z(h+0)$ is supposed to be known (initial condition). Let us denote by $X_0 := X(0)$, $X_h := X(h)$, $Z_0 := Z(0)$, and $Z_h := Z(h)$. So we obtain that $A = \frac{\gamma}{\sqrt{\gamma^2-\varepsilon_1}} Z_0$, $B = \frac{\gamma}{\sqrt{\gamma^2-\varepsilon_3}} Z_h$.

Then from conditions (11) we obtain

$$H_y^{(h)} = -Z_h \frac{\varepsilon_3}{\sqrt{\gamma^2 - \varepsilon_3}}; \quad H_y^{(0)} = Z_0 \frac{\varepsilon_1}{\sqrt{\gamma^2 - \varepsilon_1}}. \tag{12}$$

In accordance with (7) inside the layer

$$-Z'(x) + \gamma X(x) = \gamma^{-1} (\varepsilon_f + f) X(x). \tag{13}$$

Then for $x = h$ we obtain from (13)

$$-Z_h \frac{\gamma \varepsilon_3}{\sqrt{\gamma^2 - \varepsilon_3}} = \left(\varepsilon_f + f\left(X_h^2, Z_h^2\right)\right) X_h. \tag{14}$$

If $Z_h > 0$ (we assume it), then, as it is easy to see from (14), $X_h < 0$ (we used the fact that $\varepsilon_{xx} > 0$).

Denote by $f_h := f\left(X_h^2, Z_h^2\right)$ and $G_h := G\left(X_h^2, Z_h^2\right)$. Then, using first integral (10), substituting $x = h$, we find the value $C_h := C|_{x=h}$:

$$C_h = X_h^2 \left(\varepsilon_f - \gamma^2 + f_h\right)^2 + \gamma^2 \left(\left(\varepsilon_f - \gamma^2\right) X_h^2 + \varepsilon_g Z_h^2\right) + G_h. \tag{15}$$

It should be noticed that $C_h > 0$ if $Z_h > 0$ and functions f, g satisfy conditions above mentioned.

In order to find the values X_0 and Z_0 it is necessary to solve the following system:

$$\begin{cases} \frac{\gamma\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}} Z_0 = \left(\varepsilon_f + f_0\right) X_0; \\ \left(\varepsilon_f - \gamma^2 + f_0\right)^2 X_0^2 + \gamma^2 \left(\left(\varepsilon_f - \gamma^2\right) X_0^2 + \varepsilon_g Z_0^2\right) + G_0 = C_h, \end{cases} \tag{16}$$

where $f_0 = f\left(X_0^2, Z_0^2\right)$ and $G_0 = G\left(X_0^2, Z_0^2\right)$.

System (16) is obtained by using formula (13) at $x = 0$ and first integral (10) at $x = 0$.

It is easy to see from the second equation of system (16) that the values X_0 and Z_0 can have arbitrary signs. At the same time from the first equation of system (16) we can see that X_0 and Z_0 have to be positive or negative simultaneously (here the condition $\varepsilon_{xx} > 0$ is used).

Normal components of electromagnetic field are known to be discontinuous at media interfaces. And it is the discontinuity of the first kind. In this case the normal component is E_x. It is also known that the value εE_x is continuous at media interfaces. It follows from the above and from the continuity of the tangential component E_z that the transmission conditions for functions εX and Z are

$$[\varepsilon X]_{x=0} = 0, \quad [\varepsilon X]_{x=h} = 0, \quad [Z]_{x=0} = 0, \quad [Z]_{x=h} = 0, \qquad (17)$$

where $[f]_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$ denotes a jump of the function f at the interface.

We also suppose that functions $X(x)$ and $Z(x)$ satisfy the condition

$$X(x) = O\left(\frac{1}{|x|}\right) \text{ and } Z(x) = O\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty. \qquad (18)$$

Let

$$\mathrm{D} = \begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & \frac{d}{dx} \end{pmatrix}, \mathbf{F}(X,Z) = \begin{pmatrix} X \\ Z \end{pmatrix}, \mathbf{G}(\mathbf{F},\gamma) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

where $X \equiv X(x)$, $Z \equiv Z(x)$ are unknown functions; $G_1 \equiv G_1(\mathbf{F},\gamma)$, $G_2 \equiv G_2(\mathbf{F},\gamma)$ are right-hand sides of system (8). The value γ is a spectral parameter. Rewrite the problem using new notation.

For the half-space $x < 0$ and $\varepsilon = \varepsilon_1$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & \gamma \\ \frac{\gamma^2-\varepsilon_1}{\gamma} & 0 \end{pmatrix} \mathbf{F} = 0. \qquad (19)$$

Inside the layer $0 < x < h$ we have $\varepsilon_{xx} = \varepsilon_f + f\left(X^2, Z^2\right)$, $\varepsilon_{zz} = \varepsilon_g + g\left(X^2, Z^2\right)$ and we obtain

$$\mathrm{L}(\mathbf{F},\gamma) \equiv \mathrm{D}\mathbf{F} - \mathbf{G}(\mathbf{F},\gamma) = 0. \qquad (20)$$

For the half-space $x > h$ and $\varepsilon = \varepsilon_3$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & \gamma \\ \frac{\gamma^2-\varepsilon_3}{\gamma} & 0 \end{pmatrix} \mathbf{F} = 0. \qquad (21)$$

Let us formulate the transmission problem (it is possible to reformulate it as the boundary eigenvalue problem). *It is necessary to find eigenvalues* γ *and corresponding to them nonzero vectors* $\mathbf{F}$ *such that* $\mathbf{F}$ *satisfies to equations* (19)–(21). *Components* X, Z *of vector* $\mathbf{F}$ *satisfy transmission conditions* (17), *condition* (18) *and* X_0, Z_0 *satisfy to system* (16).

Definition 1. The value $\gamma = \gamma_0$ such that nonzero solution $\mathbf{F}$ to problem (19)–(21) exists under conditions (16)–(18) is called an eigenvaue of the problem. Solution $\mathbf{F}$ corresponding to the eigenvalue is called an eigenvector of the problem and components $X(x)$ and $Z(x)$ of vector $\mathbf{F}$ are called eigenfunctions (see the remark on p. 37).

§5. Dispersion Equation

Introduce the new variables[1]

$$\tau(x) = \varepsilon_f + X^2(x), \quad \eta(x) = \frac{X(x)}{Z(x)}\tau(x), \qquad (22)$$

then

$$X^2 = \tau - \varepsilon_f, \quad XZ = (\tau - \varepsilon_f)\frac{\tau}{\eta}, \quad Z^2 = (\tau - \varepsilon_f)\frac{\tau^2}{\eta^2}.$$

We also assume that here and further

$$f \equiv f\left(\tau - \varepsilon_f, (\tau - \varepsilon_f)\frac{\tau^2}{\eta^2}\right), \quad g \equiv g\left(\tau - \varepsilon_f, (\tau - \varepsilon_f)\frac{\tau^2}{\eta^2}\right).$$

The form of system (8) and first integral (10) have to be found in new variables. Logically we obtain

$$\begin{cases} \gamma\left(2X^2 f'_u + \varepsilon_f + f\right)\left(X^2\right)' = \\ \qquad = 2\left(\gamma^2\left(\varepsilon_g + g\right) + 2\left(\varepsilon_f - \gamma^2 + f\right)X^2 f'_v\right)XZ, \\ \left(Z^2\right)' = \frac{2}{\gamma}\left(\gamma^2 - \varepsilon_f - f\right)XZ, \end{cases}$$

[1]See the footnote on p. 37.

and further

$$\begin{cases} \gamma\left(2\left(\tau-\varepsilon_f\right)f'_u+\varepsilon_f+f\right)\tau' = \\ \qquad\qquad = 2\frac{\tau}{\eta}\left(\tau-\varepsilon_f\right)\left(\gamma^2\left(\varepsilon_g+g\right)+2\tau\left(\varepsilon_f-\gamma^2+f\right)f'_v\right), \\ \left(\frac{\tau^2}{\eta^2}\left(\tau-\varepsilon_f\right)\right)' = \frac{2}{\gamma}\frac{\tau}{\eta}\left(\tau-\varepsilon_f\right)\left(\gamma^2-\varepsilon_f-f\right). \end{cases}$$

From the first equation we obtain

$$\tau' = \frac{2}{\gamma}\left(\tau-\varepsilon_f\right)\frac{\tau}{\eta}\chi,$$

where $\chi = \frac{\gamma^2(\varepsilon_g+g)+2\left(\tau-\varepsilon_f\right)\left(\varepsilon_f-\gamma^2+f\right)f'_v}{2\left(\tau-\varepsilon_f\right)f'_u+\varepsilon_f+f}$.

Let us transform the second equation of the latter system

$$\frac{\tau}{\eta^2}\left(3\tau-3\varepsilon_f\right)\tau' - 2\frac{\tau^2\left(\tau-\varepsilon_f\right)}{\eta^3}\eta' = \frac{2}{\gamma}\frac{\tau}{\eta}\left(\gamma^2-\varepsilon_f-f\right)\left(\tau-\varepsilon_f\right).$$

Using τ' we obtain

$$\frac{\tau}{\eta^2}\left(\frac{1}{\gamma}\left(3\tau-3\varepsilon_f\right)\chi-\eta'\right) = \frac{1}{\gamma}\left(\gamma^2-\varepsilon_f-f\right).$$

And finally we obtain

$$\begin{cases} \tau' = \frac{2}{\gamma}\frac{\tau}{\eta}\left(\tau-\varepsilon_f\right)\chi; \\ \eta' = \frac{1}{\gamma}\frac{\eta^2}{\tau}\left(\varepsilon_f-\gamma^2+f\right)+\left(3\tau-2\varepsilon_f\right)\chi, \end{cases} \tag{23}$$

here and further

$$f'_u = \left.\frac{\partial f\left(u,v\right)}{\partial u}\right|_{\left(\tau-\varepsilon_f,\frac{\tau^2}{\eta^2}\left(\tau-\varepsilon_f\right)\right)}, \quad f'_v = \left.\frac{\partial f\left(u,v\right)}{\partial v}\right|_{\left(\tau-\varepsilon_f,\frac{\tau^2}{\eta^2}\left(\tau-\varepsilon_f\right)\right)}.$$

The first integral takes the form

$$\left(\varepsilon_f-\gamma^2+f\right)^2+\gamma^2\left(\left(\varepsilon_f-\gamma^2\right)+\varepsilon_g\frac{\tau^2}{\eta^2}\right)+ \\ + \frac{G\left(\tau-\varepsilon_f,\frac{\tau^2}{\eta^2}\left(\tau-\varepsilon_f\right)\right)}{\left(\tau-\varepsilon_f\right)} \equiv C. \tag{24}$$

Equation (24), in general, is a transcendental equation with respect to τ, η. Its solution with respect to a certain variable (τ or η) can be analytically expressed only in exceptional cases.

The functions f and g are assumed such that the right-hand side of the second equation in system (23) is positive. On the face of it, this condition seems too rigid. However it is not so. For example, if f and g are polynomials with positive coefficients, then this condition is satisfied. As it is known, polarization vector in constitutive relations in the Maxwell equations can be expanded into a series in $|\mathbf{E}|$. When we consider that nonlinearity functions are polynomials we simply cut off the series. It is necessary to remember that the condition $\frac{\partial f}{\partial\left(|E_z|^2\right)} = \frac{\partial g}{\partial\left(|E_x|^2\right)}$ constrains the forms of the polynomials f and g.

Now it is possible to find the signs of the values $\eta(0)$ and $\eta(h)$. One can see from system (16) that the values X_0 and Z_0 either positive or negative simultaneously. At the same time, from formula (14) it is clear that the values X_h and Z_h have opposite signs. Taking it into account we obtain

$$\eta\,(0) = \frac{X_0}{Z_0}\left(\varepsilon_f + X_0^2\right) > 0 \quad \text{and} \quad \eta\,(h) = \frac{X_h}{Z_h}\left(\varepsilon_f + X_0^2\right) < 0. \tag{25}$$

It is easy to see that the right-hand side of the second equation in system (23) is strictly positive. This means that the function $\eta(x)$ strictly increases on interval $(0, h)$. Taking into account (25) we obtain that the function $\eta(x)$ can not be differentiable on the entire interval $(0, h)$. This means that the function $\eta(x)$ has a break point.

Since the solutions X and Z of system (8) are analytical functions; therefore, the function η has only discontinuities of the second kind. And the points where the function Z vanishes are these discontinuities. Let the function η have a discontinuity at the point $x^* \in (0, h)$. It is obvious that in this case $\eta\,(x^* - 0) \to +\infty$ and $\eta\,(x^* + 0) \to -\infty$.

It is natural to suppose that the function $\eta(x)$ on interval $(0, h)$ has several break points $x_0,\ x_1,\ ...,\ x_N$. The properties of function $\eta(x)$ imply

$$\eta\left(x_i - 0\right) = +\infty, \quad \eta\left(x_i + 0\right) = -\infty, \ \text{где} \ \ i = \overline{0, N}. \tag{26}$$

Denote by

$$w := \left[\frac{1}{\gamma}\frac{\eta^2}{\tau}\left(\varepsilon_f - \gamma^2 + f\right) + \left(3\tau - 2\varepsilon_f\right)\chi\right]^{-1},$$

where $w = w(\eta)$; $\tau = \tau(\eta)$ is expressed from first integral (24); and χ is defined in the beginning of this section.

Taking into account our hypothesis we will seek to the solutions on each interval $[0\ , x_0)$, (x_0, x_1), ..., $(x_N\ , h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0)} w d\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} w d\eta = x + c_i, & x_i \le x \le x_{i+1}, \ \ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} w d\eta = x + c_N, & x_N \le x \le h. \end{cases} \tag{27}$$

Substituting $x = 0$, $x = x_{i+1}$, and $x = x_N$ into equations (27) (into the first, the second, and the third, respectively) and taking into account (26), we find constants $c_1,\ c_2,\ ...,\ c_{N+1}$:

$$\begin{cases} c_0 = -\int\limits_{\eta(0)}^{+\infty} w d\eta; \\ c_{i+1} = \int\limits_{-\infty}^{+\infty} w d\eta - x_{i+1}, \ \ i = \overline{0, N-1}; \\ c_{N+1} = \int\limits_{-\infty}^{\eta(h)} w d\eta - h. \end{cases} \tag{28}$$

Using (28) we can rewrite equations (27) in the following form

$$\begin{cases} \int\limits_{\eta(x)}^{\eta(x_0)} w d\eta = -x + \int\limits_{\eta(0)}^{+\infty} w d\eta, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} w d\eta = x + \int\limits_{-\infty}^{+\infty} w d\eta - x_{i+1}, & x_i \le x \le x_{i+1}, \quad i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} w d\eta = x + \int\limits_{-\infty}^{\eta(h)} w d\eta - h, & x_N \le x \le h. \end{cases} \tag{29}$$

Introduce the notation $T := \int\limits_{-\infty}^{+\infty} w d\eta$. It follows from formula (29) that $0 < x_{i+1} - x_i = T < h$, where $i = \overline{0, N-1}$. This implies the convergence of the improper integral (it will be proved in other way below). Now consider x in equations (29) such that all the integrals on the left side vanish (i.e. $x = x_0$, $x = x_i$, and $x = x_N$), and sum all equations (29). We obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{+\infty} w d\eta + x_0 + T - x_1 + ... + x_{N-1} + T - x_N + x_N + \int\limits_{-\infty}^{\eta(h)} w d\eta - h.$$

This formula implies

$$\int\limits_{\eta(0)}^{+\infty} w d\eta + \int\limits_{-\infty}^{\eta(h)} w d\eta + NT = h.$$

Finally we obtain the DE in the following form

$$-\int\limits_{\eta(h)}^{\eta(0)} w d\eta + (N+1)T = h, \tag{30}$$

where $\eta(0)$, $\eta(h)$ are defined by formulas (25).

Expression (30) is the DE, which holds for any finite h. Let γ be a solution of DE (30) and an eigenvalue of the problem. Then, there are eigenfunctions X and Z, which correspond to the eigenvalue γ. The eigenfunction Z has $N+1$ zeros on the interval $(0,h)$.

Notice that improper integrals in DE (30) converge. Indeed, function $\tau = \tau(\eta)$ is a bounded one as $\eta \to \infty$ since $\tau = \varepsilon_f + X^2$ and X are bounded functions. Then

$$|w| = \left| \frac{\gamma\tau}{\eta^2 \left(\varepsilon_f - \gamma^2 + f\right) + \gamma\tau \left(3\tau - 2\varepsilon_f\right)\chi} \right| \le \left| \frac{1}{\alpha\eta^2 + \beta} \right|,$$

where $\alpha > 0$, $\beta > 0$ are constants. It is obvious that improper integral $\int\limits_{-\infty}^{+\infty} \frac{d\eta}{\alpha\eta^2+\beta}$ converges. Convergence of the improper integrals in (30) in inner points results from the requirement that the right-hand side of the second equation of system (23) is positive.

Theorem 1. *The set of solutions of DE* (30) *contains the set of solutions* (*eigenvalues*) *of the boundary eigenvalue problem* (19)–(21) *with conditions* (16)–(18).

Proof. It follows from the method of obtaining of DE (30) from system (23) that an eigenvalue of problem (19)–(21) is a solution of the DE.

It is obvious that function τ as the function with respect to η defined from first integral (24) is a multiple-valued function. This implies that not every solution of DE (30) is an eigenvalue of the problem. In other words, system (16) can have several roots (X_0, Z_0) such that $\tau(0) \ge \varepsilon_f$. Even in this case it is possible to find eigenvalues among roots of the DE. Indeed, when we find a solution γ of DE (30), we can find functions $\tau(x)$ and $\eta(x)$ from system (23) and first integral (24). From functions $\tau(x)$ and $\eta(x)$ using formulas (22) we obtain

$$X(x) = \pm\sqrt{\tau - \varepsilon_f} \quad \text{и} \quad Z(x) = \pm\sqrt{\tau - \varepsilon_f}\,\frac{\tau}{|\eta|}. \tag{31}$$

It is an important question how to choose the signs. Let us discuss it in detail. We know the behavior of the function $\eta = \tau\frac{Y}{Z}$: it monotonically increases, and if $x = x^*$ such that $\eta\left(x^*\right) = 0$, then $\eta\left(x^* - 0\right) < 0$ and $\eta\left(x^* + 0\right) > 0$; if $x = x^{**}$ such that $\eta\left(x^{**}\right) = \pm\infty$, then $\eta\left(x^{**} - 0\right) > 0$ and $\eta\left(x^{**} + 0\right) < 0$. The function η has no other points of sign's reversal. To fix the idea, assume that the initial condition is $Z_h > 0$. If $\eta > 0$, then the functions X and Z have the same signs; if $\eta < 0$, then the functions X and Z have different signs. Since X and Z are continuous we can choose necessary signs in expressions (31). Now, when we find the function X we can calculate X_0. If this calculated value is equal to the value calculated from system (16), then the solution γ of the DE is an eigenvalue of the problem (and is not an eigenvalue otherwise).

If the functions f and g such that a unique solution (X_0, Z_0) of system (16) exists, then we have the following

Theorem 2 (*of equivalence*). *If system* (16) *has a unique solution* $(\tau(0), \eta(0))$ *and* $\tau(0) \geq \varepsilon_f$, *then boundary eigenvalue problem* (19)–(21) *with conditions* (16)–(18) *has a solution* (*an eigenvalue*) *if and only if this eigenvalue is a solution of DE* (30).

The proof of this theorem results from the proof of previous theorem.

Introduce the notation $J(\gamma, k) := \int\limits_{\eta(0)}^{\eta(h)} w d\eta + kT$, where the right-hand side is defined by DE (30) and $k = \overline{0, N+1}$.

Let

$$h_{\inf}^{k} = \inf_{\gamma^2 \in \left(\max(\varepsilon_1, \varepsilon_3), \varepsilon_f\right)} J(\gamma, k),$$
$$h_{\sup}^{k} = \sup_{\gamma^2 \in \left(\max(\varepsilon_1, \varepsilon_3), \varepsilon_f\right)} J(\gamma, k).$$

Let us formulate the sufficient condition of existence at least one eigenvalue of the problem.

Theorem 3. *Let h satisfies for a certain $k = \overline{0, N+1}$ the following two-sided inequality*

$$h_{\inf}^{k} < h < h_{\sup}^{k},$$

then boundary eigenvalue problem (19)–(21) *with conditions* (16)–(18) *has at least one solution* (*an eigenvalue*).

The quantities $h_{\inf}^{k}$ and $h_{\sup}^{k}$ can be numerically calculated.

§6. Generalized Dispersion Equation

Here we derive the generalized DE, which holds for any real values ε_{xx} and ε_{zz}. In addition the sign of the right-hand side of the second equation in system (23) (see the footnote on p. 43), and conditions $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_f$ or $0 < \gamma^2 < \varepsilon_f$ are not taken into account. These conditions appear in the case of a linear layer and are used for derivation of DE (30). Though in the nonlinear case it is not necessary to limit value γ^2 from the right side. At the same time it is clear that γ is limited from the left side, since this limit appears from the solutions in the half-spaces (where the permittivities are constants).

Now we assume that γ satisfies one of the following inequalities

$$\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < +\infty,$$

when either ε_1 or ε_3 is positive, or

$$0 < \gamma^2 < +\infty,$$

when both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$.

At first we derive the DE from system (23) and first integral (24), and then we discuss the details of derivation and conditions when the derivation is possible and the DE is well defined.

Thus, let us consider system (23) and first integral (24)

$$\begin{cases} \tau' = \frac{2}{\gamma}\frac{\tau}{\eta}\left(\tau - \varepsilon_f\right)\chi, \\ \eta' = \frac{1}{\gamma}\frac{\eta^2}{\tau}\left(\varepsilon_f - \gamma^2 + f\right) + \left(3\tau - 2\varepsilon_f\right)\chi, \end{cases}$$

where $\chi = \frac{\gamma^2(\varepsilon_g + g) + 2\left(\tau - \varepsilon_f\right)\left(\varepsilon_f - \gamma^2 + f\right) f'_v}{2\left(\tau - \varepsilon_f\right) f'_u + \varepsilon_f + f}$;

$$f'_u = \left.\frac{\partial f(u,v)}{\partial u}\right|_{\left(\tau - \varepsilon_f, \frac{\tau^2}{\eta^2}\left(\tau - \varepsilon_f\right)\right)}, \quad f'_v = \left.\frac{\partial f(u,v)}{\partial v}\right|_{\left(\tau - \varepsilon_f, \frac{\tau^2}{\eta^2}\left(\tau - \varepsilon_f\right)\right)};$$

$$\left(\varepsilon_f - \gamma^2 + f\right)^2 + \gamma^2\left(\left(\varepsilon_f - \gamma^2\right) + \varepsilon_g\frac{\tau^2}{\eta^2}\right) + \\ + G\left(\tau - \varepsilon_f, \frac{\tau^2}{\eta^2}\left(\tau - \varepsilon_f\right)\right)\left(\tau - \varepsilon_f\right)^{-1} \equiv C,$$

where $G\left(X^2, Z^2\right) \equiv \int_{Z_0}^{Z^2} g\left(X^2, s\right) ds$, C is a constant.

Using first integral (24) it is possible to integrate formally any of the equations of system (23). As earlier we integrate the second equation. We can not obtain a solution on the whole interval $(0, h)$, since function $\eta(x)$ can have break points, which belong to $(0, h)$. It is known that function $\eta(x)$ is an analytical one. Therefore we can conclude that if $\eta(x)$ has break points when $x \in (0, h)$, then there are only break points of the second kind.

Assume that the function $\eta(x)$ on interval $(0, h)$ has $N+1$ break points $x_0,\ x_1,\ ...,\ x_N$.

It should be noticed that

$$\eta(x_i - 0) = \pm\infty, \quad \eta(x_i + 0) = \pm\infty,$$

where $i = \overline{0,\ N}$, and signs $\pm$ are independent and unknown.

Taking into account the above, solutions are sought on each interval $[0\ , x_0)$, (x_0, x_1), ..., $(x_N\ , h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + c_{i+1}, & x_i \le x \le x_{i+1},\ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + c_{N+1}, & x_N \le x \le h. \end{cases} \tag{32}$$

From equations (32), substituting $x = 0$, $x = x_{i+1}$, and $x = x_N$ into the first, the second, and the third equations (32), respectively, we find required constants $c_1,\ c_2,\ ...,\ c_{N+1}$:

$$\begin{cases} c_0 = -\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta; \\ c_{i+1} = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1},\ i = \overline{0, N-1}; \\ c_{N+1} = \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h. \end{cases} \tag{33}$$

Using (33) equations (32) take the form

$$\begin{cases} \int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = -x + \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, & x_i \le x \le x_{i+1}; \\ \int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h, & x_N \le x \le h, \end{cases} \tag{34}$$

where $i = \overline{0, N-1}$.

From formulas (34) we obtain that

$$x_{i+1} - x_i = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta, \quad i = \overline{0, N-1}. \tag{35}$$

Expressions $0 < x_{i+1} - x_i < h < \infty$ imply that under the assumption about the break points existence the integral on the right-hand side converges and $\int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta > 0$. In the same way, from the first and the last equations (34) we obtain that $x_0 = \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta$ and $0 < x_0 < h$, then

$$0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta < h < \infty;$$

and $h - x_N = \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta$ and $0 < h - x_N < h$ then

$$0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta < h < \infty.$$

These considerations yield that function $\eta(x)$ has finite number of break points and function $w(\eta)$ has no nonintegrable singularities for $\eta \in (-\infty, \infty)$.

Now, setting $x = x_0$, $x = x_i$, and $x = x_N$ into the first, the second, and the third equations in (34), respectively, we have that all the integrals on the left sides vanish. We add all the equations in (34) to obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + x_0 + \int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} w d\eta - x_1 + ...$$
$$... + x_{N-1} + \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} w d\eta - x_N + x_N + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h. \quad (36)$$

From (36) we obtain

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta + \sum_{i=0}^{N-1} \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta = h. \quad (37)$$

It follows from formulas (35) that

$$\eta\left(x_i + 0\right) = \pm\infty \text{ and } \eta\left(x_i - 0\right) = \mp\infty, \text{ where } i = \overline{0, N},$$

and it is necessary to choose the infinities of different signs.

Thus we obtain that

$$\int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} w d\eta = ... = \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} w d\eta =: T'.$$

Hence $x_1 - x_0 = ... = x_N - x_{N-1}$.

Now we can rewrite equation (37) in the following form

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} f d\eta + NT' = h.$$

Let $T := \int\limits_{-\infty}^{+\infty} w d\eta$, then we finally obtain

$$-\int\limits_{\eta(h)}^{\eta(0)} w d\eta \pm (N+1)T = h, \tag{38}$$

where $\eta(0)$, $\eta(h)$ are defined by formulas (25).

Expression (38) is the DE, which holds for any finite h. Let γ be a solution of DE (38) and an eigenvalue of the problem. Then, there are eigenfunction X and Z, which correspond to the eigenvalue γ. The eigenfunction Z has $N+1$ zeros on the interval $(0, h)$. It should be noticed that for every number $N+1$ it is necessary to solve two DEs: for $N+1$ and for $-(N+1)$.

Let us formulate the following

Theorem 4. *The set of solutions of DE* (38) *contains the set of solutions* (*eigenvalues*) *of boundary eigenvalue problem* (19)–(21) *with conditions* (16)–(18).

The proof of this theorem is almost word-by-word coincides to the proof of theorem 1.

Now, let us review some theoretical treatments of the derivation of DEs (30) and (38). We are going to discuss the existence and uniqueness of system's (8) solutions.

Let us consider vector form (20) of system (8)

$$\mathrm{D}\mathbf{F} = \mathbf{G}(\mathbf{F}, \lambda). \tag{39}$$

Let the right-hand side $\mathbf{G}$ be defined and continuous in the domain $\Omega \subset \mathbb{R}^2$, $\mathbf{G} : \Omega \to \mathbb{R}^2$. Also we suppose that $\mathbf{G}$ satisfies the Lipschitz condition on $\mathbf{F}$ (locally in Ω)[1].

Under these conditions system (8) (or system (39)) has a unique solution in the domain Ω [8, 41, 22].

It is clear that under these conditions system (23) has a unique solution (of course, the domain of uniqueness Ω' for variables τ, η differs from Ω).

[1]About the Lipschitz condition see the footnote on p. 48.

Since we seek bounded solutions X and Z; therefore we obtain

$$\Omega \subset [-m_1, m_1] \times [-m_2, m_2],$$

where

$$\max_{x\in[0,h]} |Y| < m_1, \quad \max_{x\in[0,h]} |Z| < m_2,$$

and the previous implies that

$$\Omega' \subset [\varepsilon_f, \varepsilon_f + m_1^2] \times (-\infty, +\infty).$$

It is easy to show that there is no point $x^* \in \Omega'$ such that $X|_{x=x^*} = 0$ and $Z|_{x=x^*} = 0$. Indeed, it is known from theory of autonomous system (see, for example, [41]) that phase trajectories do not intersect one another in the system's phase space when right-hand side of the system is continuous and satisfies the Lipschitz condition. Since $X \equiv 0$ and $Z \equiv 0$ are stationary solutions of system (8), it is obvious that nonconstant solutions X and Z can not vanish simultaneously at a certain point $x^* \in \Omega'$ (otherwise the nonconstant solutions intersect with the stationary solutions and we obtain a contradiction).

Note 1. If there is a certain value γ_*^2, such that some of the integrals in DEs (30) or (38) diverge at a certain inner points, then this simply means that the value γ_*^2 is not a solution of chosen DE and the value γ_*^2 is not an eigenvalue of the problem.

Note 2. This problem depends on the initial condition Z_h, see the note on p. 49 for further details.

It should be also noticed that if the first integral is an algebraic function (with respect to any of its variable), then the solutions of system (8) are Abelian functions[1] [6, 54, 36].

We derived the DEs from the second equation of system (23). It is possible to do it using the first equation of the system (see p. 130).

[1] In the case of TE waves and generalized Kerr nonlinearity in a layer solutions are expressed in terms of an elliptic function (see Ch. 4). In the case of TM waves and Kerr nonlinearity in a layer solutions are expressed in terms of hyperelliptic functions [6, 13, 42] . Hyperelliptic functions are closely connected with Jacobi inverse problem [13, 15].

CHAPTER 7

TM WAVE PROPAGATION IN AN ISOTROPIC LAYER WITH KERR NONLINEARITY

§1. Statement of the Problem

Let us consider electromagnetic waves propagating through a homogeneous isotropic nonmagnetic dielectric layer. The layer is located between two half-spaces: $x < 0$ and $x > h$ in Cartesian coordinate system $Oxyz$. The half-spaces are filled with isotropic nonmagnetic media without any sources and characterized by permittivities $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, respectively, where ε_0 is the permittivity of free space[1]. Assume that everywhere $\mu = \mu_0$, where μ_0 is the permeability of free space.

The electromagnetic field depends on time harmonically [17]

$$\tilde{\mathbf{E}}(x,y,z,t) = \mathbf{E}_+(x,y,z)\cos\omega t + \mathbf{E}_-(x,y,z)\sin\omega t,$$
$$\tilde{\mathbf{H}}(x,y,z,t) = \mathbf{H}_+(x,y,z)\cos\omega t + \mathbf{H}_-(x,y,z)\sin\omega t,$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_+,\ \mathbf{E}_-,\ \tilde{\mathbf{H}},\ \mathbf{H}_+,\ \mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-.$$

where

$$\mathbf{E} = (E_x, E_y, E_z)^T, \quad \mathbf{H} = (H_x, H_y, H_z)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

[1] Generally, conditions $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$ are not necessary. They are not used for derivation of DEs, but they are useful for DEs' solvability analysis.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential field components on the media interfaces $x = 0$, $x = h$ and the radiation condition at infinity: the electromagnetic field exponentially decays as $|x| \to \infty$ in the domains $x < 0$ and $x > h$.

The permittivity inside the layer is described by Kerr law

$$\varepsilon = \varepsilon_2 + a\,|\mathbf{E}|^2,$$

where a and $\varepsilon_2 > \max(\varepsilon_1, \varepsilon_3)$ are positive constants[1].

The solutions to the Maxwell equations are sought in the entire space.

The geometry of the problem is shown in Fig. 1.

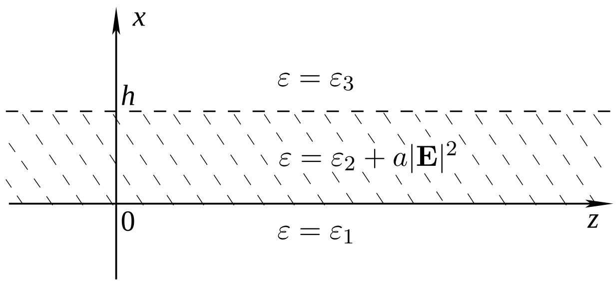


Fig. 1.

§2. TM Waves

Let us consider TM waves

$$\mathbf{E} = (E_x, 0, E_z)^T, \quad \mathbf{H} = (0, H_y, 0)^T,$$

where $E_x = E_x(x, y, z)$, $E_z = E_z(x, y, z)$, and $H_y = H_y(x, y, z)$.

[1] In §6 the solutions are sought under more general conditions.

Substituting the fields into Maxwell equations (1) we obtain

$$\begin{cases} \frac{\partial E_z}{\partial y} = 0, \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega\mu H_y, \\ \frac{\partial E_x}{\partial y} = 0, \\ \frac{\partial H_y}{\partial z} = i\omega\varepsilon E_x, \\ \frac{\partial H_y}{\partial x} = -i\omega\varepsilon E_z. \end{cases}$$

It is obvious from the first and the third equations of this system that $E_z = E_z(x,z)$ and $E_x = E_x(x,z)$ do not depend on y. This implies that H_y does not depend on y.

Waves propagating along medium interface z depend on z harmonically. This means that the fields components have the form

$$E_x = E_x(x)e^{i\gamma z},\ E_z = E_z(x)e^{i\gamma z},\ H_y = H_y(x)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system [17]

$$\begin{cases} i\gamma E_x(x) - E_z'(x) = i\omega\mu H_y(x), \\ H_y'(x) = -i\omega\varepsilon E_z(x), \\ i\gamma H_y(x) = i\omega\varepsilon E_x(x), \end{cases} \tag{2}$$

where $(\,\cdot\,)' \equiv \frac{d}{dx}$.

The following equation can be easily derived from the previous system

$$H_y(x) = \frac{1}{i\omega\mu}\left(i\gamma E_x(x) - E_z'(x)\right). \tag{3}$$

Differentiating equation (3) and using the second and the third equations of system (2) we obtain

$$\begin{cases} \gamma\left(iE_x(x)\right)' - E_z''(x) = \omega^2\mu\varepsilon E_z(x), \\ \gamma^2\left(iE_x(x)\right) - \gamma E_z'(x) = \omega^2\mu\varepsilon\left(iE_x(x)\right). \end{cases} \tag{4}$$

Let us denote by $k_0^2 := \omega^2\mu_0\varepsilon_0$ and perform the normalization according to the formulas $\tilde{x} = k_0 x$, $\frac{d}{dx} = k_0\frac{d}{d\tilde{x}}$, $\tilde{\gamma} = \frac{\gamma}{k_0}$, $\tilde{\varepsilon}_j = \frac{\varepsilon_j}{\varepsilon_0}$

($j = 1, 2, 3$), $\tilde{a} = \frac{a}{\varepsilon_0}$. Denoting by $Z(\tilde{x}) := E_z$, $X(\tilde{x}) := iE_x$ and omitting the tilde symbol, from system (4) we obtain

$$\begin{cases} -Z'' + \gamma X' = \varepsilon Z, \\ -Z' + \gamma X = \frac{1}{\gamma}\varepsilon X. \end{cases} \tag{5}$$

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along boundaries of the layer $0 < x < h$, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of the spectral parameter γ such that real solutions $X(x)$ and $Z(x)$ to system (5) exist (so $|\mathbf{E}|^2$ does not depend on z, see the footnote on p. 33 and the note on p. 74). We consider that

$$\varepsilon = \begin{cases} \varepsilon_1, & x < 0; \\ \varepsilon_2 + a\left(X^2 + Z^2\right), & 0 < x < h; \\ \varepsilon_3, & x > h. \end{cases} \tag{6}$$

Also we assume that $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$. This two-sided inequality naturally appears for an analogous problem in a layer with a constant permittivity tensor (for further details see Ch. 4, formula (14)).

Also we assume that functions X and Z are sufficiently smooth

$$\begin{aligned} X(x) \in & C\,(-\infty,\ 0] \cap C[0,h] \cap C\,[h,\ +\infty) \cap \\ & \cap C^1\,(-\infty,\ 0] \cap C^1[0,h] \cap C^1\,[h,\ +\infty)\,; \\ Z(x) \in & C(-\infty,+\infty) \cap C^1\,(-\infty,\ 0] \cap C^1[0,h] \cap C^1\,[h,\ +\infty) \cap \\ & \cap C^2(-\infty,0) \cap C^2(0,h) \cap C^2(h,+\infty). \end{aligned}$$

Physical nature of the problem implies these conditions.

It is clear that system (5) is an autonomous one. System (5) can be rewritten in a normal form (it will be done below). This system in the normal form can be considered as a dynamical system with analytical with respect to X and Z right-hand sides[1]. It is

[1] Of course, in the domain where these right-hand sides are analytical with respect to X and Z.

well known (see, for example [5]) that the solution X and Z of such a system are analytical functions with respect to the independent variable as well. This is an important fact for DEs' derivation.

We consider that γ satisfies the inequality $\gamma^2 > \max(\varepsilon_1, \varepsilon_3)$. This condition occurs in the case if at least one of the values ε_1 or ε_3 is positive. If both values ε_1 and ε_3 are negative, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem

In the domain $x < 0$ we have $\varepsilon = \varepsilon_1$. From system (5) we obtain $X' = \gamma Z$, $Z' = \frac{\gamma^2 - \varepsilon_1}{\gamma} X$. It implies the equation $X'' = (\gamma^2 - \varepsilon_1)X$. Its general solution is

$$X(x) = A_1 e^{-x\sqrt{\gamma^2-\varepsilon_1}} + A e^{x\sqrt{\gamma^2-\varepsilon_1}}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} X(x) &= A \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right), \\ Z(x) &= \frac{\sqrt{\gamma^2-\varepsilon_1}}{\gamma} A \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right). \end{aligned} \tag{7}$$

We assume that $\gamma^2 - \varepsilon_1 > 0$ itherwise it will be impossible to satisfy the radiation condition.

In the domain $x > h$ we have $\varepsilon = \varepsilon_3$. From system (5) we obtain $X' = \gamma Z$, $Z' = \frac{\gamma^2 - \varepsilon_3}{\gamma} X$. It implies the equation $X'' = (\gamma^2 - \varepsilon_3)X$. Its general solution is

$$X(x) = B e^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}} + B_1 e^{(x-h)\sqrt{\gamma^2-\varepsilon_3}}.$$

In accordance with the radiation condition we obtain

$$\begin{aligned} X(x) &= B \exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right), \\ Z(x) &= -\frac{\sqrt{\gamma^2-\varepsilon_3}}{\gamma} B \exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right). \end{aligned} \tag{8}$$

Here for the same reason as above we consider that $\gamma^2 - \varepsilon_3 > 0$.

Constants A and B in (7) and (8) are defined by transmission conditions and initial conditions.

Inside the layer $0 < x < h$ system (5) takes the form

$$\begin{cases} -\frac{d^2 Z}{dx^2} + \gamma \frac{dX}{dx} = \left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right) Z, \\ -\frac{dZ}{dx} + \gamma X = \frac{1}{\gamma}\left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right) X. \end{cases} \tag{9}$$

Differentiating the second equation we obtain

$$-Z'' + \gamma X' = \frac{2a}{\gamma}(XX' + ZZ')X + \frac{1}{\gamma}\left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right) X'.$$

Using this equation system (9) can be rewritten in the following form[1]

$$\begin{cases} \frac{dX}{dx} = \frac{2a\left(\varepsilon_2 - \gamma^2 + a\left(X^2+Z^2\right)\right)X^2 + \gamma^2\left(\varepsilon_2 + a\left(X^2+Z^2\right)\right)}{\gamma\left(\varepsilon_2 + 3aX^2 + aZ^2\right)} Z, \\ \frac{dZ}{dx} = -\frac{1}{\gamma}\left(\varepsilon_2 - \gamma^2 + a\left(X^2 + Z^2\right)\right) X. \end{cases} \tag{10}$$

Dividing the first equation in system (10) to the second one we obtain the ordinary differential equation

$$-\left(\varepsilon_2 + 3aX^2 + aZ^2\right)\frac{dX}{dZ} = \\ = 2aXZ + \gamma^2 \frac{\varepsilon_2 + a\left(X^2 + Z^2\right)}{\varepsilon_2 - \gamma^2 + a\left(X^2 + Z^2\right)} \frac{Z}{X}. \tag{11}$$

Equation (11) can be transformed into a total differential equation. Indeed, rewrite it into a symmetric form

$$MdX + NdZ = 0,$$

where

$$M = \left(\varepsilon_2 + 3aX^2 + aZ^2\right)\left(\varepsilon_2 - \gamma^2 + aX^2 + aZ^2\right) X,$$
$$N = \left(2a\left(\varepsilon_2 - \gamma^2 + aX^2 + aZ^2\right) X^2 + \gamma^2\left(\varepsilon_2 + aX^2 + aZ^2\right)\right) Z.$$

It is easy to check that the relation $\frac{\partial M}{\partial Z} = \frac{\partial N}{\partial X}$ is satisfied. This means that equation (11) can be rewritten as a total differential

[1] Now system (10) is written in a normal form. If the right-hand sides are analytic functions with respect to X and Z, then the solutions are analytic functions with respect to its independent variable. We notice it in the end of §2.

equation (the equation $MdX + NdZ = 0$ is the total differential equation). Let us find its solution $U(X,Z)$ (it is the first integral of system (10)). Rewrite M in the following form

$$M = \left(\varepsilon_2 + aZ^2\right)\left(\varepsilon_2 + aZ^2 - \gamma^2\right)X + \\ + 3aX^3\left(\varepsilon_2 + aZ^2 - \gamma^2\right) + aX^3\left(\varepsilon_2 + aZ^2\right) + 3a^2X^5.$$

Since $\frac{\partial U}{\partial x} = M$; therefore, we obtain

$$U(X,Z) = \int MdX = \frac{1}{2}\left(\varepsilon_2 + aZ^2\right)\left(\varepsilon_2 + aZ^2 - \gamma^2\right)X^2 + \\ + \frac{3a}{4}X^4\left(\varepsilon_2 + aZ^2 - \gamma^2\right) + \frac{a}{4}X^4\left(\varepsilon_2 + aZ^2\right) + \frac{a^2}{2}X^6 + \varphi(Z).$$

Let us find $\varphi(Z)$ from the equation $\frac{\partial U}{\partial Z} = N$

$$aX^2Z\left(\varepsilon_2 + aZ^2 - \gamma^2\right) + aX^2Z\left(\varepsilon_2 + aZ^2\right) + \\ + \frac{3a^2}{2}X^4Z + \frac{a^2}{2}X^4Z + \varphi'(Z) = N.$$

From this equation we obtain $\varphi'(Z) = \gamma^2\varepsilon_2 Z + \gamma^2 aZ^3$. This implies

$$\varphi(Z) = \frac{\gamma^2\varepsilon_2}{2}Z^2 + \frac{\gamma^2 a}{4}Z^4.$$

Taking into account derived results the first integral can be written in the following form

$$2\left(aZ^2 + \varepsilon_2\right)\left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)\left(2\gamma^2 - \left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)\right) = \\ = \gamma^6 C + 3\gamma^2\left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)^2 - 2\left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)^3, \quad (12)$$

where C is a constant of integration.

§4. Transmission Conditions and the Transmission Problem

Tangential components of an electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are H_y and E_z. Hence, we obtain

$$\begin{aligned} &H_y\left(h+0\right)=H_y\left(h-0\right), \quad H_y\left(0-0\right)=H_y\left(0+0\right), \\ &E_z\left(h+0\right)=E_z\left(h-0\right), \quad E_z\left(0-0\right)=E_z\left(0+0\right). \end{aligned}$$

From the continuity conditions for the tangential components of the fields **E** and **H** we obtain

$$\begin{aligned} &\gamma X(h)-Z'(h)=H_y^{(h)}, \qquad \gamma X(0)-Z'(0)=H_y^{(0)}, \\ &Z(h)=E_z(h+0)=E_z^{(h)}, \quad Z(0)=E_z(0-0)=E_z^{(0)}, \end{aligned} \tag{13}$$

where $H_y^{(h)} := i\frac{\sqrt{\mu}}{\sqrt{\varepsilon_0}}H_y(h+0)$, $H_y^{(0)} := i\frac{\sqrt{\mu}}{\sqrt{\varepsilon_0}}H_y(0-0)$.

The constant $E_z^{(h)} := E_z(h+0)$ is supposed to be known (initial condition). Let us denote by $X_0 := X(0)$, $X_h := X(h)$, $Z_0 := Z(0)$, and $Z_h := Z(h)$. So we obtain that $A = \frac{\gamma}{\sqrt{\gamma^2-\varepsilon_1}}Z_0$, $B = \frac{\gamma}{\sqrt{\gamma^2-\varepsilon_3}}Z_h$.

Then from conditions (13) we obtain

$$H_y^{(h)} = -Z_h\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}, \quad H_y^{(0)} = Z_0\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}. \tag{14}$$

In accordance with (5), (6) inside the layer

$$-Z'(x)+\gamma X(x) = \frac{1}{\gamma}\left(\varepsilon_2 + a\left(X^2(x)+Z^2(x)\right)\right)X(x). \tag{15}$$

Then for $x = h$, using (13), we obtain from (15)

$$\frac{1}{\gamma}\left(\varepsilon_2 + a\left(X_h^2+Z_h^2\right)\right)X_h = H_y^{(h)}. \tag{16}$$

From (16) we obtain the equation with respect to X_h:

$$X_h^3 + \frac{\varepsilon_2 + aZ_h^2}{a}X_h - \frac{\gamma H_y^{(h)}}{a} = 0. \tag{17}$$

Under taken assumptions (in regard to ε_2 and a) the value $\frac{\varepsilon_2 + aZ_h^2}{a} > 0$. Hence, equation (17) has at least one real root, which is considered

$$X_h = \left(\frac{\gamma H_y^{(h)}}{2a} + \sqrt{\frac{1}{27}\left(\frac{\varepsilon_2}{a} + Z_h^2\right)^3 + \frac{1}{4}\left(\frac{\gamma}{a}\right)^2 \left(H_y^{(h)}\right)^2} \right)^{1/3} + \\ + \left(\frac{\gamma H_y^{(h)}}{2a} - \sqrt{\frac{1}{27}\left(\frac{\varepsilon_2}{a} + Z_h^2\right)^3 + \frac{1}{4}\left(\frac{\gamma}{a}\right)^2 \left(H_y^{(h)}\right)^2} \right)^{1/3}.$$

Using first integral (12) at $x = h$, we find the value $C_h^X := C|_{x=h}$ from the equation

$$2\left(aZ_h^2 + \varepsilon_2\right)\left(\varepsilon_2 + a\left(X_h^2 + Z_h^2\right)\right)\left(2\gamma^2 - \left(\varepsilon_2 + a\left(X_h^2 + Z_h^2\right)\right)\right) = \\ = \gamma^6 C_h^X + 3\gamma^2\left(\varepsilon_2 + a\left(X_h^2 + Z_h^2\right)\right)^2 - 2\left(\varepsilon_2 + a\left(X_h^2 + Z_h^2\right)\right)^3. \tag{18}$$

In order to find the values X_0 and Z_0 it is necessary to solve the following system[1]

$$\begin{cases} \frac{\gamma\varepsilon_1}{\sqrt{\gamma^2 - \varepsilon_1}} Z_0 = \left(\varepsilon_2 + a\left(X_0^2 + Z_0^2\right)\right) X_0, \\ 2\left(aZ_0^2 + \varepsilon_2\right)\left(\varepsilon_2 + a\left(X_0^2 + Z_0^2\right)\right)\left(2\gamma^2 - \left(\varepsilon_2 + a\left(X_0^2 + Z_0^2\right)\right)\right) = \\ \qquad = \gamma^6 C_h^X + 3\gamma^2\left(\varepsilon_2 + a\left(X_0^2 + Z_0^2\right)\right)^2 - 2\left(\varepsilon_2 + a\left(X_0^2 + Z_0^2\right)\right)^3. \end{cases} \tag{19}$$

It is easy to see from the second equation of system (19) that the values X_0 and Z_0 can have arbitrary signs. At the same time from the first equation of this system we can see that X_0 and Z_0 have to be positive or negative simultaneously.

Normal components of electromagnetic field are known to be discontinues at media interfaces. And it is the discontinuity of the first kind. In this case the normal component is E_x. It is also known that the value εE_x is continuous at media interfaces. From the above

[1]This system is obtained using formula (15) at $x = 0$ and the first integral at the same point.

and from the continuity of the tangential component E_z it follows that the transmission conditions for the functions εX and Z are

$$[\varepsilon X]_{x=0} = 0, \quad [\varepsilon X]_{x=h} = 0, \quad [Z]_{x=0} = 0, \quad [Z]_{x=h} = 0, \qquad (20)$$

where $[f]_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$ denotes a jump of the function f at the interface.

We also suppose that functions $X(x)$ and $Z(x)$ satisfy the condition

$$X(x) = O\left(\frac{1}{|x|}\right) \quad \text{and} \quad Z(x) = O\left(\frac{1}{|x|}\right) \quad \text{as} \quad |x| \to \infty. \qquad (21)$$

Let

$$\mathrm{D} = \begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & \frac{d}{dx} \end{pmatrix}, \mathbf{F}(X,Z) = \begin{pmatrix} X \\ Z \end{pmatrix}, \mathbf{G}(\mathbf{F},\gamma) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

where $X \equiv X(x)$, $Z \equiv Z(x)$ are unknown functions; $G_1 \equiv G_1(\mathbf{F},\gamma)$, $G_2 \equiv G_2(\mathbf{F},\gamma)$ are right-hand sides of system (10). The value γ is a spectral parameter. Rewrite the problem using new notations.

For the half-space $x < 0$ and $\varepsilon = \varepsilon_1$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & \gamma \\ \frac{\gamma^2 - \varepsilon_1}{\gamma} & 0 \end{pmatrix} \mathbf{F} = 0. \qquad (22)$$

Inside the layer $0 < x < h$ and $\varepsilon = \varepsilon_2 + a\left(X^2 + Z^2\right)$, we have

$$\mathrm{L}(\mathbf{F},\gamma) \equiv \mathrm{D}\mathbf{F} - \mathbf{G}(\mathbf{F},\gamma) = 0. \qquad (23)$$

For the half-space $x > h$ and $\varepsilon = \varepsilon_3$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & \gamma \\ \frac{\gamma^2 - \varepsilon_3}{\gamma} & 0 \end{pmatrix} \mathbf{F} = 0. \qquad (24)$$

Let us formulate the transmission problem (it is possible to reformulate it as the boundary eigenvalue problem). *It is necessary to find eigenvalues* γ *and corresponding to them nonzero vectors* $\mathbf{F}$ *such that* $\mathbf{F}$ *satisfies to equations* (22)–(24). *Components* X, Z *of vector* $\mathbf{F}$ *satisfy transmission conditions* (20), *condition* (21) *and* X_0, Z_0 *satisfy to system* (19).

Definition 1. The value $\gamma = \gamma_0$ such that nonzero solution $\mathbf{F}$ to problem (22)–(24) exists under conditions (19)–(21) is called an eigenvalue of the problem. Solution $\mathbf{F}$ corresponding to the eigenvalue is called an eigenvector of the problem, and components $X(x)$ and $Z(x)$ of the vector $\mathbf{F}$ are called eigenfunctions (see the note on p. 37).

§5. Dispersion Equation

Introduce the new variables

$$\tau(x) = \frac{\varepsilon_2 + a\left(X^2(x) + Z^2(x)\right)}{\gamma^2}, \quad \eta(x) = \gamma \frac{X(x)}{Z(x)}\tau(x). \tag{25}$$

Let $\tau_0 = \frac{\varepsilon_2}{\gamma^2}$, then

$$X^2 = \frac{\gamma^2}{a}\frac{\eta^2(\tau-\tau_0)}{\eta^2+\gamma^2\tau^2},\ Z^2 = \frac{\gamma^4}{a}\frac{\tau^2(\tau-\tau_0)}{\eta^2+\gamma^2\tau^2},\ XZ = \frac{\gamma^3}{a}\frac{\tau\eta(\tau-\tau_0)}{\eta^2+\gamma^2\tau^2}.$$

Using new variables rewrite system (10) and equation (12)

$$\begin{cases} \frac{d\tau}{dx} = 2\gamma^2 \frac{\tau^2\eta(\tau-\tau_0)(2-\tau)}{\tau(\eta^2+\gamma^2\tau^2)+2\eta^2(\tau-\tau_0)}, \\ \frac{d\eta}{dx} = \frac{\gamma^2\tau^2+\eta^2(\tau-1)}{\tau}, \end{cases} \tag{26}$$

$$\eta^2 = \frac{\gamma^2\tau^2\left(\tau^2 - C\right)}{C + 3\tau^2 - 2\tau^3 - 2\tau(2-\tau)\tau_0}, \tag{27}$$

the constant C is not equal to the value of the same name in (12).

Equation (27) is an algebraic quartic one with respect to τ. Its solution $\tau = \tau(\eta)$ can be expressed in explicit form using Cardanus formulas [28].

In order to obtain the DE for the propagation constants it is necessary to find the values $\eta(0)$, $\eta(h)$.

It is clear that $\eta(0) = \gamma\frac{X(0)}{Z(0)}\tau(0)$, $\eta(h) = \gamma\frac{X(h)}{Z(h)}\tau(h)$. Taking into account that $X(x)\tau(x) = \varepsilon X(x)$ and using formulas (13), (14), it is easy to obtain that

$$\eta(0) = \frac{\varepsilon_1}{\sqrt{\gamma^2 - \varepsilon_1}} > 0, \quad \eta(h) = -\frac{\varepsilon_3}{\sqrt{\gamma^2 - \varepsilon_3}} < 0. \tag{28}$$

The value C (we denote it as C_h^τ) can be easily find from first integral (27). Indeed, the values $\tau(h)$ and $\eta(h)$ are known, using first integral (27) at $x = h$ we obtain

$$C_h^\tau = \tau^2(h) - \frac{2\varepsilon_3^2\tau(h)(2 - \tau(h))(\tau(h) - \tau_0)}{\varepsilon_3^2 + \gamma^2(\gamma^2 - \varepsilon_3)\tau^2(h)}, \tag{29}$$

where $\tau(h) = \frac{H_y^{(h)}}{\gamma X_h}$.

If $C_h^\tau > 0$, then equation (27) with respect to $\tau(h)$ has a positive root. It is easy to prove that C_h^τ is strictly greater than zero. Indeed, it is easy to see from expression (29) that if $\tau(h) > 2$, then the value $C_h^\tau > 0$, as $\tau(h) \geq \tau_0 > 1$ and $\left(\gamma^2 - \varepsilon_3\right) > 0$. Let $\tau(h) \in [\tau_0, 2)$. Reduce to a common denominator expression (29) and replace (if it is necessary) $\tau(h) = \tau_0 + \alpha$, where $0 < \alpha < 1$ we obtain the following expression

$$C_h^\tau = \tau(h)\frac{\gamma^2\left(\gamma^2 - \varepsilon_3\right)\tau^3(h) + \varepsilon_3^2(2\alpha(\tau(h) - 1) + \tau_0 - \alpha)}{\varepsilon_3^2 + \gamma^2(\gamma^2 - \varepsilon_3)\tau^2(h)}$$

with strictly positive right-hand side.

It is easy to see that the right-hand side of the second equation of system (26) is strictly positive. This means that the function $\eta(x)$ monotonically increases on interval $(0, h)$. Taking into account (28) we obtain that the function $\eta(x)$ can not be differentiable on the entire interval $(0, h)$. This means that the function $\eta(x)$ has a break point. Let $x^* \in (0, h)$ be the break point. From (27) it is obvious that x^* is such that $\tau^* = \tau(x^*)$ is a root of the equation $C_h^\tau + 3(\tau^*)^2 - 2(\tau^*)^3 - 2\tau^*(2 - \tau^*)\tau_0 = 0$. In addition $\eta(x^* - 0) \to +\infty$ and $\eta(x^* + 0) \to -\infty$.

It is natural to suppose that the function $\eta(x)$ on interval $(0, h)$ has several break points $x_0,\ x_1,\ ...,\ x_N$. The properties of function $\eta(x)$ imply

$$\eta(x_i - 0) = +\infty,\ \eta(x_i + 0) = -\infty,\ \text{где } i = \overline{0, N}. \tag{30}$$

Let

$$w := \frac{\tau}{\gamma^2\tau^2 + \eta^2\,(\tau - 1)},$$

where $w = w(\eta)$; $\tau = \tau(\eta)$ is expressed from equation (15).

Taking into account our hypothesis we will seek to the solutions on each interval $[0\ , x_0)$, (x_0, x_1), ..., $(x_N\ , h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0)} w d\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} w d\eta = x + c_i, & x_i \le x \le x_{i+1},\ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} w d\eta = x + c_N, & x_N \le x \le h. \end{cases} \tag{31}$$

Substituting $x = 0$, $x = x_{i+1}$, and $x = x_N$ into equations (31) (into the first, the second, and the third, respectively) and taking into account (30), we find constants c_1, c_2, $\ldots$, c_{N+1}:

$$\begin{cases} c_0 = -\int\limits_{\eta(0)}^{+\infty} w d\eta; \\ c_{i+1} = \int\limits_{-\infty}^{+\infty} w d\eta - x_{i+1},\ i = \overline{0, N-1}; \\ c_{N+1} = \int\limits_{-\infty}^{\eta(h)} w d\eta - h. \end{cases} \tag{32}$$

Using (32) we can rewrite equations (31) in the following form

$$\begin{cases} \int\limits_{\eta(x)}^{\eta(x_0)} w d\eta = -x + \int\limits_{\eta(0)}^{+\infty} w d\eta, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} w d\eta = x + \int\limits_{-\infty}^{+\infty} w d\eta - x_{i+1}, & x_i \le x \le x_{i+1},\ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} w d\eta = x + \int\limits_{-\infty}^{\eta(h)} w d\eta - h, & x_N \le x \le h. \end{cases} \tag{33}$$

Introduce the notation $T := \int\limits_{-\infty}^{+\infty} w d\eta$. It follows from formula (33) that $0 < x_{i+1} - x_i = T < h$, where $i = \overline{0, N-1}$. This implies the convergence of the improper integral (it will be proved in other way below). Now consider x in equations (33) such that all the integrals on the left side vanish (i.e. $x = x_0$, $x = x_i$, and $x = x_N$), and sum all equations (33). We obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{+\infty} w d\eta + x_0 + T - x_1 + ... + x_{N-1} + T - x_N + x_N + \int\limits_{-\infty}^{\eta(h)} w d\eta - h.$$

Finally we obtain

$$-\int\limits_{-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}}^{\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}} w d\eta + (N+1)T = h. \tag{34}$$

Expression (34) is the DE, which holds for any finite h. Let γ be a solution of DE (34) and an eigenvalue of the problem. Then, there are eigenfunctions X and Z, which correspond to the eigenvalue γ. The eigenfunction Z has $N+1$ zeros on the interval $(0, h)$.

Notice that improper integrals in DE (34) converge. Indeed, function $\tau = \tau(\eta)$ is bounded as $\eta \to \infty$ since $\tau = \frac{\varepsilon_2 + aX^2 + aZ^2}{\gamma^2}$, and X, Z are bounded.

Then

$$|w| = \left| \frac{\tau}{\gamma^2\tau^2 + \eta^2(\tau - 1)} \right| \leq \frac{1}{\alpha\eta^2 + \beta},$$

where $\alpha > 0$, $\beta > 0$ are constants. It is obvious that improper integral $\int\limits_{-\infty}^{\infty} \frac{d\eta}{\alpha\eta^2+\beta}$ converges. Convergence of the improper integrals in (34) in inner points results from the requirement that the right-hand side of the second equation of system (26) is positive.

The first equation of system (26) jointly with the first integral can be integrated in hyperelliptic functions. The solution is expressed in implicit form by means of hyperelliptic integrals. This is the simple example of Abelian integrals. The inversion of these integrals are hyperelliptic functions and they are solutions of system (26). Hyperelliptic functions are Abelian functions, which are meromorphic and periodic functions. Since function η is expressed algebraically through τ; therefore, η is a meromorphic periodic function. This means that the break point x^* is a pole of function η. The integral in equation (34) is a more general Abelian integral [6, 36].

Theorem 1 (*of equivalence*). *Boundary eigenvalue problem* (22)–(24) *with conditions* (19)–(21) *has a solution* (*an eigenvalue*) *if and only if this eigenvalue is a solution of DE* (34).

Proof. *Sufficiency*. It is obvious that if we find the solution γ of DE (34), then we can find functions $\tau(x)$ and $\eta(x)$ from system (26) and first integral (27). From functions $\tau(x)$ and $\eta(x)$, and using formulas (25) we find

$$X(x) = \pm\frac{\gamma}{\sqrt{a}}\eta\sqrt{\frac{\tau - \tau_0}{\eta^2 + \gamma^2\tau^2}} \text{ and } Z(x) = \pm\frac{\gamma^2}{\sqrt{a}}\tau\sqrt{\frac{\tau - \tau_0}{\eta^2 + \gamma^2\tau^2}}. \tag{35}$$

It is an important question how to choose the signs. Let us discuss it in detail. We know the behavior of the function $\eta = \gamma\tau\frac{X}{Z}$: it monotonically increases, and if $x = x^*$ such that $\eta(x^*) = 0$, then $\eta(x^* - 0) < 0$, $\eta(x^* + 0) > 0$; if $x = x^{**}$ such that $\eta(x^{**}) = \pm\infty$, then $\eta(x^{**} - 0) > 0$ and $\eta(x^{**} + 0) < 0$. Function η has no other points of sign's reversal. To fix the idea, assume that the initial condition is $Z(h) > 0$. If $\eta > 0$, then the functions X and Z have the same signs; if $\eta < 0$, the functions X and Z have different signs. Since X and Z are continuous functions[1] we can choose necessary signs in expressions (35).

Necessity. It follows from the method of obtaining of DE (34) from system (26) that an eigenvalue of the problem is a solution of the DE.

It should be also noticed that eigenfunctions (or eigenmodes) that correspond an eigenvalue γ_0 can be easily numerically calculated from system (9) or (10), (for example, using a Runge-Kutta method).

Introduce the notation $J(\gamma, k) := \int\limits_{\eta(0)}^{\eta(h)} w d\eta + kT$, where the right-hand side is defined by DE (34) and $k = \overline{0, N+1}$.

Let

$$h^k_{\inf} = \inf_{\gamma^2 \in (\max(\varepsilon_1, \varepsilon_3), \varepsilon_2)} J(\gamma, k),$$

$$h^k_{\sup} = \sup_{\gamma^2 \in (\max(\varepsilon_1, \varepsilon_3), \varepsilon_2)} J(\gamma, k).$$

Let us formulate the sufficient condition of existence at least one eigenvalue of the theorem.

Theorem 2. *Let h satisfies for a certain $k = \overline{0, N+1}$ the following two-sided inequality*

$$h^k_{\inf} < h < h^k_{\sup},$$

then boundary eigenvalue problem (22)–(24) *with conditions* (19)–(21) *has at least one solution* (*an eigenvalue*).

The quantities $h^k_{\inf}$ and $h^k_{\sup}$ can be numerically calculated.

§6. Generalized Dispersion Equation

Here we derive the generalized DE, which holds for any real values ε_2. In addition the sign of the right-hand side of the second equation in system (26) (see the footnote on p. 43), and conditions

[1]Of course, we mean that X and Z are continuous in the regions $x < 0$, $0 < x < h$, and $x > h$.

$\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$ or $0 < \gamma^2 < \varepsilon_2$ are not taken into account. These conditions appear in the case of a linear layer and are used for derivation of DE (34). Though on the nonlinear case it is not necessary to limit the value γ^2 from the right side. At the same time it is clear that γ is limited from the left side, since this limit appears from the solutions in the half-spaces.

Now we assume that γ satisfies one of the following inequalities

$$\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < +\infty,$$

when either ε_1 or ε_3 is positive, or

$$0 < \gamma^2 < +\infty,$$

when both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$.

At first we derive the DE from system (26) and first integral (27). After this we discuss the details of the derivation and conditions when the derivation is possible and the DE is well defined.

Thus, let us consider system (26) and first integral (27)

$$\begin{cases} \frac{d\tau}{dx} = 2\gamma^2 \frac{\tau^2\eta(\tau-\tau_0)(2-\tau)}{\tau(\eta^2+\gamma^2\tau^2)+2\eta^2(\tau-\tau_0)}, \\ \frac{d\eta}{dx} = \frac{\gamma^2\tau^2+\eta^2(\tau-1)}{\tau}; \end{cases}$$

and $\eta^2 = \frac{\gamma^2\tau^2\left(\tau^2-C\right)}{C+3\tau^2-2\tau^3-2\tau(2-\tau)\tau_0}$.

Using first integral (27) it is possible to integrate formally any of the equations of system (26). As earlier we integrate the second equation. We can not obtain the solution on the entire interval $(0, h)$, since function $\eta(x)$ can have break points, which belong to $(0, h)$. It is known that function $\eta(x)$ has break points only of the second kind (η is an analytical function).

Assume that function $\eta(x)$ on interval $(0, h)$ has $N+1$ break points x_0, x_1, ..., x_N.

It should be noticed that

$$\eta(x_i - 0) = \pm\infty \quad \eta(x_i + 0) = \pm\infty,$$

where $i = \overline{0, N}$, and signs $\pm$ are independent and unknown.

Taking into account the above, solutions are sought on each interval $[0\ , x_0)$, (x_0, x_1), ..., $(x_N\ , h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + c_{i+1}, & x_i \le x \le x_{i+1},\ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + c_{N+1}, & x_N \le x \le h. \end{cases} \quad (36)$$

From equations (36), substituting $x = 0$, $x = x_{i+1}$, and $x = x_N$ into the first, the second, and the third equations (36), respectively, we find required constants $c_1,\ c_2,\ ...,\ c_{N+1}$:

$$\begin{cases} c_0 = -\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta; \\ c_{i+1} = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1},\ i = \overline{0, N-1}; \\ c_{N+1} = \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h. \end{cases} \quad (37)$$

Using (37) equations (36) take the form

$$\begin{cases} \int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = -x + \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, & x_i \le x \le x_{i+1}; \\ \int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h, & x_N \le x \le h, \end{cases} \quad (38)$$

where $i = \overline{0, N-1}$.

From formulas (38) we obtain that

$$x_{i+1} - x_i = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta,\ i = \overline{0, N-1}. \quad (39)$$

Expressions $0 < x_{i+1} - x_i < h < \infty$ imply that under the assumption about the break point existence the integral on the right side converges and $\int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta > 0$. In the same way, from the first and the last equations (38) we obtain that $x_0 = \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta$ and $0 < x_0 < h$ then

$$0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta < h < \infty;$$

and $h - x_N = \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta$ and $0 < h - x_N < h$then

$$0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta < h < \infty.$$

These considerations yield that the function $w(\eta)$ has no non-integrable singularities for $\eta \in (-\infty, \infty)$. And also this proves that the assumption about a finite number break points is true.

Now, setting $x = x_0$, $x = x_i$, and $x = x_N$ into the first, the second, and the third equations in (38), respectively, we have that all the integrals on the left sides vanish. We add all the equations in (38) to obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + x_0 + \int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} w d\eta - x_1 + ...$$
$$... + x_{N-1} + \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} w d\eta - x_N + x_N + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h. \quad (40)$$

From (40) we obtain

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta + \sum_{i=0}^{N-1} \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta = h. \tag{41}$$

It follows from formulas (39) that

$$\eta\left(x_i+0\right) = \pm\infty \text{ and } \eta\left(x_i-0\right) = \mp\infty, \text{ where } i = \overline{0, N},$$

and it is necessary to choose the infinities of different signs.

Thus we obtain that

$$\int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} w d\eta = ... = \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} w d\eta =: T'.$$

Hence $x_1 - x_0 = ... = x_N - x_{N-1}$.

Now we can rewrite equation (41) in the following form

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} f d\eta + NT' = h.$$

Let $T \equiv \int\limits_{-\infty}^{+\infty} w d\eta$, the we finally obtain

$$-\int\limits_{\eta(h)}^{\eta(0)} w d\eta \pm (N+1)T = h, \tag{42}$$

where $\eta(0)$, $\eta(h)$ are defined by formulas (28).

Expression (42) is the DE, which holds for any finite h. Let γ be a solution of DE (42) and an eigenvalue of the problem. Then, there are eigenfunctions X and Z, which correspond to the eigenvalue γ. The eigenfunction Z has $N+1$ zeros on the interval $(0, h)$. It should be noticed that for every number $N+1$ it is necessary to solve two DEs: for $N+1$ and for $-(N+1)$.

Let us formulate the following

Theorem 3. *The set of solutions of DE* (42) *contains the set of solutions* (*eigenvalues*) *of boundary eigenvalue problem* (22)–(24) *with conditions* (19)–(21).

Proof. It is obvious that this theorem generalises Theorem 1. It is also obvious that any eigenvalue of the problem is a solution of the DE. It is easy to understand where additional solutions of the DE occur from (the solutions, which are not eigenvalues). If the values ε_2 and a are arbitrary real values, then equation (17) and system (19) can have several roots. And it is possible that among these roots we can not choose roots that correspond to the problem. In other words, for each group of three roots we have DE (42). It is clear that not all the solutions of these DEs are eigenvalues of the problem. A solution of the DE is an eigenvalue of the problem if and only if transmission conditions (20) are satisfied. That is, if we have a solution γ of the DE, on the one hand, then we can find X_0, Z_0, and X_h. On the other hand, we can find the values X_0, Z_0, and X_h from equation (17) and system (19). The solution γ is an eigenvalue if and only if each value found in one way coincides with corresponding value found in other way. Using this criterion we can determine eigenvalues among solutions of the DE. This criterion can be easily used for numerical calculation.

Now, let us review some theoretical treatments of derivation of DEs (34) and (42). We are going to discuss the existence and uniqueness of system's (10) solutions.

Let us consider vector form (23) of system (10)

$$\mathrm{D}\mathbf{F} = \mathbf{G}(\mathbf{F}, \lambda). \tag{43}$$

Let the right-hand side $\mathbf{G}$ be defined and continuous in the domain $\Omega \subset \mathbb{R}^2$, $\mathbf{G} : \Omega \to \mathbb{R}^2$. Also we suppose that $\mathbf{G}$ satisfies the Lipschitz condition on $\mathbf{F}$ (locally in Ω)[1].

Under these conditions system (10) (or system (43)) has a unique solution in the domain Ω [8, 41, 22].

[1]About the Lipschitz condition see the footnote on p. 48.

It is clear that under these conditions system (26) has a unique solution (of course, the domain of uniqueness Ω' for variables τ, η differs from Ω).

Since we seek bounded solutions X and Z; therefore we obtain

$$\Omega \subset [-m_1, m_1] \times [-m_2, m_2],$$

where

$$\max_{x\in[0,h]} |Y| < m_1, \quad \max_{x\in[0,h]} |Z| < m_2,$$

and the previous implies that

$$\Omega' \subset [\varepsilon_f, \varepsilon_f + m_1^2] \times (-\infty, +\infty).$$

It is easy to show that there is no point $x^* \in \Omega'$, such that $X|_{x=x^*} = 0$ and $Z|_{x=x^*} = 0$. Indeed, it is known from theory of autonomous system (see, for example, [41]) that phase trajectories do not intersect one another in the system's phase space when right-hand side of the system is continuous and satisfies the Lipschitz condition. Since $\tilde{X} \equiv 0$ and $\tilde{Z} \equiv 0$ are stationary solutions of system (10), it is obvious that the nonconstant solutions X and Z can not vanish simultaneously at a certain point $x^* \in \Omega'$ (otherwise the nonconstant solutions intersect with the stationary solutions and we obtain a contradiction).

Note 1. If there is a certain value γ_*^2, such that some of the integrals in DEs (34) or (42) diverge at certain inner points, then this simply means that the value γ_*^2 is not a solution of chosen DE and the value γ_*^2 is not an eigenvalue of the problem.

Note 2. This problem depends on the initial condition Z_h, see the note on p. 49 for further details.

We derived the DEs from the second equation of system (26). It is possible to do it using the first equation of the system.

The DE obtained from the first equation of system (26) and first integral (27) is given below. We omit the derivation of the DE. At first the DE was derived under conditions $\varepsilon_2 > \max(\varepsilon_1, \varepsilon_3)$ and $a > 0$. It is rather easy to generalise the DE for arbitrary real values ε_2 and a. Though we do not do it and below it will be quite understandable why.

The DE has the form

$$-\int\limits_{+\sqrt{C_h^\tau}}^{\tau(0)} g d\tau + \int\limits_{\tau(h)}^{+\sqrt{C_h^\tau}} g d\tau + 2(N+1) \int\limits_{+\sqrt{C_h^\tau}}^{\tau^*} g d\tau = h, \qquad (*)$$

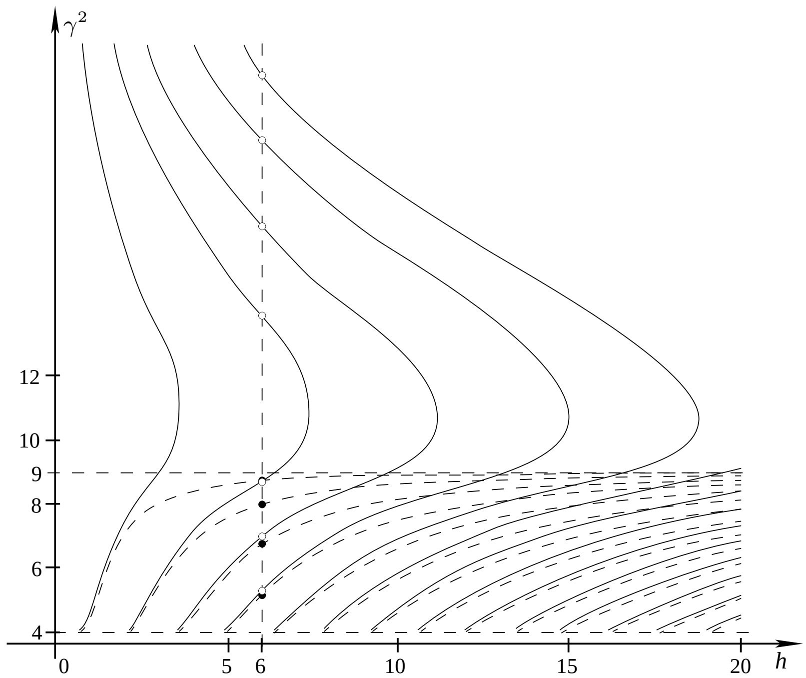
where $g = +\frac{\gamma\tau\sqrt{\tau^2 - C_h^\tau}}{\sqrt{C_h^\tau + 3\tau^2 - 2\tau^3 - 2\tau(2-\tau)\tau_0}}$; τ^* is such that $\eta(\tau^*) = \pm\infty$; C_h^τ is defined by formula (29); $\tau(h) = \frac{H_y^{(h)}}{\gamma X_h}$ and $\tau(0)$ is defined as a root of the equation

$$\zeta^4 + \frac{2\varepsilon_1^2}{\gamma^2\left(\gamma^2 - \varepsilon_1\right)}\zeta^3 - \left(\frac{\varepsilon_1^2\left(3\gamma^2 + 2\varepsilon_2\right)}{\gamma^4\left(\gamma^2 - \varepsilon_1\right)} + C_h^\tau\right)\zeta^2 + \\ + \frac{2\varepsilon_1^2\varepsilon_2}{\gamma^4\left(\gamma^2 - \varepsilon_1\right)}\zeta - \frac{\varepsilon_1^2 C_h^\tau}{\gamma^2\left(\gamma^2 - \varepsilon_1\right)} = 0.$$

It is obvious that the limits of integration in DE $(*)$ are defined rather complicated. In spite of the fact that the integrand is simpler than the one in DE (34) it is more convenient to use (particularly for calculation) the DE obtained from the second equation of system (26). This is the reason why we do not give the derivation of DE $(*)$. DE $(*)$ is given here for demonstration the fact that it is possible to use the first equation of system (26).

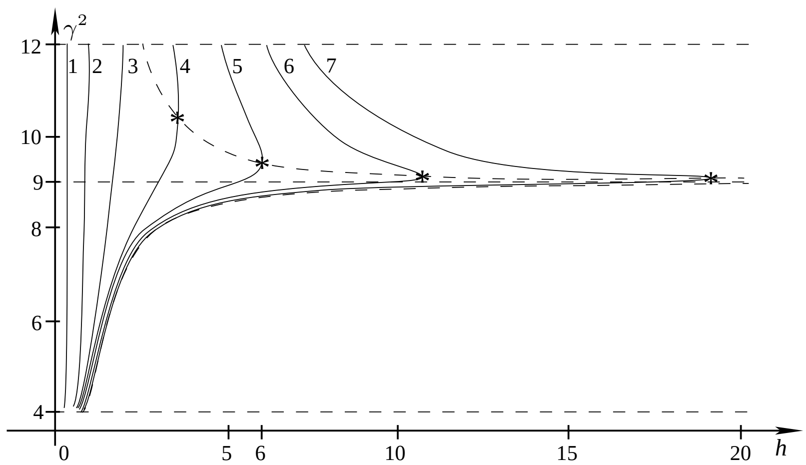
Dispersion curves (DC) calculated from equation (42) are shown in Fig. 2, 3.

DCs for both linear and nonlinear cases are shown in Fig. 2. Solid curves denote the solutions of DE (42); dashed curves denote the solution of equation (42) when $a = 0$, i.e. the solution of the DE for linear medium in the layer (see (19), Ch. 5 or formula (44) in this chapter). The following parameters are used $\varepsilon_1 = 4$, $\varepsilon_2 = 9$, $\varepsilon_3 = 1$ (these parameters are applied to both linear and nonlinear cases); in addition for the nonlinear case the values $a = 0.1$ (nonlinearity coefficient) and $Z_h = 1$ (initial condition) are used. Dashed lines are described by formulas $h = 6$ (thickness of the layer), $\gamma^2 = 4$ (lower bound for γ^2), $\gamma^2 = 9$ (upper bound for γ^2 in the case of linear medium in the layer).



γ^2
12
10
9
8
6
4
0
5
6
10
15
20
h

Fig. 2



γ^2
12
10
9
8
6
4
1
2
3
4
5
6
7
0
5
6
10
15
20
h

Fig. 3

As it is known (see Ch. 5) and it is shown in Fig. 2, the line $\gamma^2 = 9$ is an asymptote for DCs in the linear case. It should be noticed that in the linear case in the region $\gamma^2 \geq \varepsilon_2$ DCs are absent. It can be proved that function $h \equiv h(\gamma)$ defined from equation (42) is continuous at the neighborhood $\gamma^2 = \varepsilon_2$ when $a \neq 0$ (see Fig. 2). This is the important distinction between linear and nonlinear cases.

Further, it can be proved that function $h \equiv h(\gamma)$ defined from equation (42) when $a \neq 0$ has the following property:

$$\lim_{\gamma^2 \to +\infty} h(\gamma) = 0.$$

In Fig. 2 for $h = 6$ in the case of a linear layer there are 4 eigenvalues (black dots where the line $h = 6$ intersects DCs). These eigenvalues correspond to 4 eigenmodes. In the case of a nonlinear layer in Fig. 2 are shown 7 eigenvalues (uncolored dots). These eigenvalues correspond to 7 eigenmodes. Taking into account the last paragraph's statement it is clear that in this case there is infinite number of eigenvalues. What is more, the sequence $\{\gamma_i\}_{i=1}^{\infty}$ of the eigenvalues is an unbounded monotonically increasing sequence. And the sequence of layer's thicknesses $\{h_i\}_{i=1}^{\infty}$, which correspond to the sequence $\{\gamma_i\}_{i=1}^{\infty}$ is a bounded by zero monotonically decreasing sequence.

In Fig. 3 are shown DCs for the nonlinear layer with different values of nonlinearity coefficient a. The solid curves denote the solution of DE (42); the dashed curve denotes the solutions of equation (42) when $a = 0$, i.e. the solutions of the DE for linear medium in the layer (see (19), Ch. 5 or formula (44) in this chapter). The following parameters are used: $\varepsilon_1 = 4$, $\varepsilon_2 = 9$, $\varepsilon_3 = 1$ (these parameters have to do with both linear and nonlinear cases); in addition for the nonlinear case the value $Z_h = 1$ (initial condition) is used.

In Fig. 3 for DCs (solid curves) 1–7 the following values of nonlinearity coefficient a are used: $1 - a = 100$; $2 - a = 10$; $3 - a = 1$; $4 - a = 0.1$; $5 - a = 0.01$; $6 - a = 0.001$; $7 - a = 0.0001$. The DC for the linear case is not almost seen so closely it fits to the DC for the nonlinear layer ($a = 0.0001$).

It is easy to see in Fig. 3 that the less nonlinearity coefficient a the more stretched DCs in the nonlinear case. The maximum points of the curves $h(\gamma)$ (in Fig. 3 they are marked by asterisks) move to the right. The parts of the DCs that locate below the maximum points asymptotically tend to the DCs for the linear case as $a \to 0$.

§7. Passage to the Limit in the Generalized Dispersion Equation

Let us consider the passage to the limit as $a \to 0$. The value $a = 0$ corresponds to the case of a linear medium in the layer. Here the two cases are possible:

a) $\varepsilon_2 > 0$;

b) $\varepsilon_2 < 0$ (metamaterial case).

Let us examine the (a) case. The DE for a linear case is well known [64] and has the form

$$\operatorname{tg}\left(h\sqrt{\varepsilon_2-\gamma^2}\right) = \frac{\varepsilon_2\sqrt{\varepsilon_2-\gamma^2}\left(\varepsilon_1\sqrt{\gamma^2-\varepsilon_3}+\varepsilon_3\sqrt{\gamma^2-\varepsilon_1}\right)}{\varepsilon_1\varepsilon_3\left(\varepsilon_2-\gamma^2\right)-\varepsilon_2^2\sqrt{\gamma^2-\varepsilon_3}\sqrt{\gamma^2-\varepsilon_1}}. \quad (44)$$

Let

$$f = \frac{\tau}{\gamma^2\tau^2+\eta^2\left(\tau-1\right)}, \quad f_1 = \frac{\varepsilon_2}{\varepsilon_2-\gamma^2}\frac{1}{\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2}+\eta^2}.$$

Using passage to the limit as $a \to 0$ we obtain the function f_1 from the function f. We seek bounded solutions $X(x)$ and $Z(x)$. This implies that the denominator of the function f_1 can not vanish. What is more, the function f as $a \to 0$ tends to the function f_1 uniformly on $x \in [0, h]$. It is possible to pass the limit under integral sign as $a \to 0$ in (42) using results of classical analysis

$$h = \frac{\varepsilon_2}{\varepsilon_2-\gamma^2}\times$$
$$\times\left(-\int\limits_{-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}}^{\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}}\frac{1}{\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2}+\eta^2}d\eta+(N+1)\int\limits_{-\infty}^{+\infty}\frac{1}{\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2}+\eta^2}d\eta\right). \quad (45)$$

The integrals in (45) are calculated analytically. Calculating these integrals we obtain

$$h\sqrt{\varepsilon_2-\gamma^2}= \\ =\operatorname{arctg}\frac{\varepsilon_2\sqrt{\varepsilon_2-\gamma^2}\left(\varepsilon_1\sqrt{\gamma^2-\varepsilon_3}+\varepsilon_3\sqrt{\gamma^2-\varepsilon_1}\right)}{\varepsilon_1\varepsilon_3\left(\varepsilon_2-\gamma^2\right)-\varepsilon_2^2\sqrt{\gamma^2-\varepsilon_3}\sqrt{\gamma^2-\varepsilon_1}}+(N+1)\,\pi. \tag{46}$$

Expression (46) can be easily transformed into expression (44).

Let us examine the (b) case. We have $\varepsilon_2<0$ (metamaterial) and the DE for the linear case has the form (the derivation see in Ch. 5):

$$e^{2h\sqrt{\gamma^2-\varepsilon_2}}= \\ =\frac{\varepsilon_1\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_1}}{\varepsilon_1\sqrt{\gamma^2-\varepsilon_2}+\varepsilon_2\sqrt{\gamma^2-\varepsilon_1}}\cdot\frac{\varepsilon_3\sqrt{\gamma^2-\varepsilon_2}-\varepsilon_2\sqrt{\gamma^2-\varepsilon_3}}{\varepsilon_3\sqrt{\gamma^2-\varepsilon_2}+\varepsilon_2\sqrt{\gamma^2-\varepsilon_3}}, \tag{47}$$

where $\gamma^2-\varepsilon_1>0$, $\gamma^2-\varepsilon_2>0$, and $\gamma^2-\varepsilon_3>0$.

In the same way as above, passing to the limit in the function f as $a\to 0$ we obtain $f_2=\frac{|\varepsilon_2|}{\gamma^2-\varepsilon_2}\frac{1}{\eta^2-\frac{\varepsilon_2^2}{\gamma^2-\varepsilon_2}}$. Passing to the limit in equation (42) as $a\to 0$ and integrating the function f_2 we obtain

$$2h\sqrt{\gamma^2-\varepsilon_2}= \\ =-\ln\left|\frac{\eta-\frac{|\varepsilon_2|}{\gamma^2-\varepsilon_2}}{\eta+\frac{|\varepsilon_2|}{\gamma^2-\varepsilon_2}}\right|\Bigg|_{-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}}^{\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}}+(N+1)\ln\left|\frac{\eta-\frac{|\varepsilon_2|}{\gamma^2-\varepsilon_2}}{\eta+\frac{|\varepsilon_2|}{\gamma^2-\varepsilon_2}}\right|\Bigg|_{-\infty}^{\infty}.$$

It is obviously that the multiplier behind $(N+1)$ is equal to 0. Done simple calculation we obtain formula (47).

The results of this section show that it is possible to pass to the limit as $a\to 0$. DE (42) for the nonlinear case turn into equation (44) or (47) for the linear case as $a\to 0$.

§8. First Approximation for Eigenvalues of the Problem

Let

$$F\left(a,\gamma\right) = h, \tag{48}$$

where $F(a,\gamma)$ is the left-hand side of equation (42).

Expression (48) defines the implicit function $\gamma \equiv \gamma(a)$. Let us assume that it is a differentiable function at the neighborhood $a = 0$ (below we shall prove that it is really so). Expand it into Taylor series

$$\gamma \equiv \gamma\left(a\right) = \gamma\left(0\right) + \left.\frac{d\gamma\left(a\right)}{da}\right|_{a=0} a + O\left(a^2\right) = \gamma_0 + \gamma_1 a + O\left(a^2\right), \tag{49}$$

where γ_0 is a solution to the equation (44).

Calculating the total differential of equation (48) and expressing the required derivative we obtain

$$\frac{d\gamma\left(a\right)}{da} = -\frac{\frac{\partial F(a,\gamma)}{\partial a}}{\frac{\partial F(a,\gamma)}{\partial \gamma}}. \tag{50}$$

Using (34) we find

$$\frac{\partial F\left(a,\gamma\right)}{\partial a} = -\int\limits_{\eta(h)}^{\eta(0)} \frac{\partial G\left(a,\gamma,\eta\right)}{\partial a} d\eta + \left(N+1\right)\int\limits_{-\infty}^{+\infty} \frac{\partial G\left(a,\gamma,\eta\right)}{\partial a} d\eta \tag{51}$$

and

$$\frac{\partial F\left(a,\gamma\right)}{\partial \gamma} = G\left(a,\gamma,\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}\right)\frac{\gamma\varepsilon_1}{\sqrt{\left(\gamma^2-\varepsilon_1\right)^3}} + G\left(a,\gamma,-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}\right)\frac{\gamma\varepsilon_3}{\sqrt{\left(\gamma^2-\varepsilon_3\right)^3}} - \int\limits_{\eta(h)}^{\eta(0)} \frac{\partial G\left(a,\gamma,\eta\right)}{\partial \gamma} d\eta + \left(N+1\right)\int\limits_{-\infty}^{+\infty} \frac{\partial G\left(a,\gamma,\eta\right)}{\partial \gamma} d\eta, \tag{52}$$

where
$$G\left(a,\gamma,\eta\right)=\frac{\tau}{\gamma^2\tau^2+\eta^2\left(\tau-1\right)}, \tag{53}$$
$\eta(0)=\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}$, $\eta(h)=-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}$ (see formulas (28)).

In formula (53) the value τ is a function with respect to η and it is defined from equation (27).

It can be proved that $\frac{\partial G(a,\gamma,\eta)}{\partial a}$ and $\frac{\partial G(a,\gamma,\eta)}{\partial \gamma}$ tend to the functions $\left.\frac{\partial G(a,\gamma,\eta)}{\partial a}\right|_{a=0}=G_1(\gamma,\eta)$ and $\left.\frac{\partial G(a,\gamma,\eta)}{\partial \gamma}\right|_{a=0}=G_2(\gamma,\eta)$, respectively, uniformly on $x\in[0,h]$ as $a\to 0$. Assuming that the functions $\frac{\partial G(a,\gamma,\eta)}{\partial a}$ and $\frac{\partial G(a,\gamma,\eta)}{\partial \gamma}$ are continuous with respect to η under any fixed value a and using results of classical analysis it is possible to pass to the limit under the integral sign. Then formulas (51) and (52) take the forms

$$\left.\frac{\partial F\left(a,\gamma\right)}{\partial a}\right|_{a=0}=-\int\limits_{\eta(h)}^{\eta(0)}G_1\left(\gamma,\eta\right)d\eta+\left(N+1\right)\int\limits_{-\infty}^{+\infty}G_1\left(\gamma,\eta\right)d\eta, \tag{54}$$

$$\begin{aligned}\left.\frac{\partial F\left(a,\gamma\right)}{\partial \gamma}\right|_{a=0}=&\,G\left(0,\gamma,\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}\right)\frac{\gamma\varepsilon_1}{\sqrt{\left(\gamma^2-\varepsilon_1\right)^3}}+\\&+G\left(0,\gamma,-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}\right)\frac{\gamma\varepsilon_3}{\sqrt{\left(\gamma^2-\varepsilon_3\right)^3}}-\\&-\int\limits_{\eta(h)}^{\eta(0)}G_2\left(\gamma,\eta\right)d\eta+\left(N+1\right)\int\limits_{-\infty}^{+\infty}G_2\left(\gamma,\eta\right)d\eta,\end{aligned} \tag{55}$$

where
$$G\left(0,\gamma,\eta\right)=\frac{\varepsilon_2}{\varepsilon_2-\gamma^2}\frac{1}{\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2}+\eta^2}, \tag{56}$$
$\eta(0)=\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}$, $\eta(h)=-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}$ (see formulas (28)).

Using (53) we find

$$\frac{\partial G\left(a,\gamma,\eta\right)}{\partial a}=-\frac{\partial\tau}{\partial a}\frac{\gamma^2\tau^2+\eta^2}{\left(\gamma^2\tau^2+\eta^2\left(\tau-1\right)\right)^2}, \tag{57}$$

$$\frac{\partial G\left(a,\gamma,\eta\right)}{\partial\gamma}=-\frac{\partial\tau}{\partial\gamma}\frac{\gamma^2\tau^2+\eta^2}{\left(\gamma^2\tau^2+\eta^2\left(\tau-1\right)\right)^2}-\frac{2\gamma\tau^3}{\left(\gamma^2\tau^2+\eta^2\left(\tau-1\right)\right)^2}. \tag{58}$$

Pass to the limit as $a\to 0$ from formula (27) we obtain

$$\left.\frac{\partial\tau}{\partial a}\right|_{a=0}=\frac{\left(\gamma^2\tau_0^2+\eta^2\right)}{2\tau_0\left(\gamma^2\tau_0^2+\eta^2\left(\tau_0-1\right)\right)}\left(\left.\frac{\partial C_h^\tau}{\partial a}\right|_{a=0}\right). \tag{59}$$

Using (29) and passing to the limit as $a\to 0$ we obtain

$$\left.\frac{\partial C_h^\tau}{\partial a}\right|_{a=0}=2\frac{\varepsilon_2}{\gamma^2}\frac{\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)+\varepsilon_3^2\left(\varepsilon_2-\gamma^2\right)}{\gamma^2\varepsilon_3^2+\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)}\left(\left.\frac{\partial\tau\left(h\right)}{\partial a}\right|_{a=0}\right). \tag{60}$$

Using $\tau(h)=\frac{H_y^{(h)}}{\gamma X_h}$ and passing to the limit as $a\to 0$ we obtain

$$\left.\tau\left(h\right)\right|_{a=0}=\frac{\varepsilon_2}{\gamma^2};\quad \left.\frac{\partial\tau\left(h\right)}{\partial a}\right|_{a=0}=\frac{\gamma^2\varepsilon_3^2+\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)}{\gamma^2\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)}Z_h^2. \tag{61}$$

Taking into account (61) we can finally calculate (60)

$$\left.\frac{\partial C_h^\tau}{\partial a}\right|_{a=0}=2\frac{\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)+\varepsilon_3^2\left(\varepsilon_2-\gamma^2\right)}{\gamma^4\varepsilon_2\left(\gamma^2-\varepsilon_3\right)}Z_h^2. \tag{62}$$

From expressions (29) and (61) it is clear that

$$\left.C_h^\tau\right|_{a=0}=\left(\frac{\varepsilon_2}{\gamma^2}\right)^2. \tag{63}$$

Further, from (25) as $a\to 0$ we find

$$\left.\frac{\partial\tau}{\partial\gamma}\right|_{a=0}=-2\frac{\varepsilon_2}{\gamma^3}. \tag{64}$$

With the help of (56) we can calculate values required in (55)

$$\begin{aligned} &G\left(0,\gamma,\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}\right)\frac{\gamma\varepsilon_1}{\sqrt{\left(\gamma^2-\varepsilon_1\right)^3}}= \\ &\qquad =\frac{\gamma\varepsilon_1\varepsilon_2}{\sqrt{\gamma^2-\varepsilon_1}\left(\varepsilon_2^2\left(\gamma^2-\varepsilon_1\right)+\varepsilon_1^2\left(\varepsilon_2-\gamma^2\right)\right)}, \\ &G\left(0,\gamma,-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}\right)\frac{\gamma\varepsilon_3}{\sqrt{\left(\gamma^2-\varepsilon_3\right)^3}}= \\ &\qquad =\frac{\gamma\varepsilon_2\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}\left(\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)+\varepsilon_3^2\left(\varepsilon_2-\gamma^2\right)\right)}. \end{aligned} \tag{65}$$

Now we can find explicit expressions for the functions $G_1(\gamma,\eta)$ and $G_2(\gamma,\eta)$ from formulas (54) and (55), respectively. Using (57), (59), and (62) we obtain

$$G_1(\gamma,\eta)=-k\frac{\left(\frac{\varepsilon_2^2}{\gamma^2}+\eta^2\right)^2}{\left(\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2}+\eta^2\right)^3}, \tag{66}$$

where $k=\gamma^4\frac{\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)+\varepsilon_3^2\left(\varepsilon_2-\gamma^2\right)}{\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)\left(\varepsilon_2-\gamma^2\right)^3}Z_h^2$; using (58) and (64) we obtain

$$G_2(\gamma,\eta)=\frac{2\gamma\varepsilon_2}{\left(\varepsilon_2-\gamma^2\right)^2}\frac{\eta^2}{\left(\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2}+\eta^2\right)^2}. \tag{67}$$

With the help of expressions (66) and (67) we can write required derivative (50) in the following form

$$\gamma_1\equiv\left.\frac{d\gamma\left(a\right)}{da}\right|_{a=0}=\frac{\gamma^3\left(\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)+\varepsilon_3^2\left(\varepsilon_2-\gamma^2\right)\right)Z_h^2}{2\varepsilon_2^3\left(\gamma^2-\varepsilon_3\right)\left(\varepsilon_2-\gamma^2\right)}\frac{P\left(\gamma\right)}{Q\left(\gamma\right)}, \tag{68}$$

where

$$P=-\int\limits_{\eta(h)}^{\eta(0)}\frac{\left(\frac{\varepsilon_2^2}{\gamma^2}+\eta^2\right)^2}{\left(\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2}+\eta^2\right)^3}d\eta+(N+1)\int\limits_{-\infty}^{+\infty}\frac{\left(\frac{\varepsilon_2^2}{\gamma^2}+\eta^2\right)^2}{\left(\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2}+\eta^2\right)^3}d\eta, \tag{69}$$

and

$$Q = -\int\limits_{\eta(h)}^{\eta(0)} \frac{\eta^2}{\left(\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2} + \eta^2\right)^2} d\eta + (N+1) \int\limits_{-\infty}^{+\infty} \frac{\eta^2}{\left(\frac{\varepsilon_2^2}{\varepsilon_2-\gamma^2} + \eta^2\right)^2} d\eta +$$
$$+ \frac{\left(\varepsilon_2 - \gamma^2\right)^2}{2} \frac{\varepsilon_1}{\sqrt{\gamma^2 - \varepsilon_1}\left(\varepsilon_2^2\left(\gamma^2 - \varepsilon_1\right) + \varepsilon_1^2\left(\varepsilon_2 - \gamma^2\right)\right)} +$$
$$+ \frac{\left(\varepsilon_2 - \gamma^2\right)^2}{2} \frac{\varepsilon_3}{\sqrt{\gamma^2 - \varepsilon_3}\left(\varepsilon_2^2\left(\gamma^2 - \varepsilon_3\right) + \varepsilon_3^2\left(\varepsilon_2 - \gamma^2\right)\right)}, \quad (70)$$

$\eta(0) = \frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}$, $\eta(h) = -\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}$ (see formulas (28)).

It is clear from formulas (68)–(70) that under conditions for ε_1, ε_2, ε_3, γ, and a (see §1) derivative (50) is positive.

The integral in (69) and (70) are calculated elementary. Calculating the integrals and using (where it is necessary) (45) we obtain the required derivative in the following form

$$\gamma_1 \equiv \left.\frac{d\gamma(a)}{da}\right|_{a=0} = \frac{\left(\varepsilon_2^2\left(\gamma^2 - \varepsilon_3\right) + \varepsilon_3^2\left(\varepsilon_2 - \gamma^2\right)\right) Z_h^2}{8\gamma\varepsilon_2^3\left(\varepsilon_2 - \gamma^2\right)\left(\gamma^2 - \varepsilon_3\right)} \frac{P_1}{Q_1}, \quad (71)$$

where

$$P_1 = \left(2\gamma^2 - \varepsilon_2\right)\left(2\varepsilon_2^2 k_1 + \left(3\varepsilon_2 + 2\gamma^2\right) k_2\right) +$$
$$+ \frac{3\varepsilon_2^2 - 4\gamma^2\varepsilon_2 + 4\gamma^4}{\varepsilon_2} h \quad (72)$$

and

$$Q_1 = \frac{\varepsilon_1\left(\varepsilon_2 - \varepsilon_1\right)}{\sqrt{\gamma^2 - \varepsilon_1}\left(\varepsilon_2^2\left(\gamma^2 - \varepsilon_1\right) + \varepsilon_1^2\left(\varepsilon_2 - \gamma^2\right)\right)} +$$
$$+ \frac{\varepsilon_3\left(\varepsilon_2 - \varepsilon_3\right)}{\sqrt{\gamma^2 - \varepsilon_3}\left(\varepsilon_2^2\left(\gamma^2 - \varepsilon_3\right) + \varepsilon_3^2\left(\varepsilon_2 - \gamma^2\right)\right)} + \frac{h}{\varepsilon_2}, \quad (73)$$

where

$$k_1 = \frac{\varepsilon_1\sqrt{\left(\gamma^2-\varepsilon_1\right)^3}}{\left(\varepsilon_2^2\left(\gamma^2-\varepsilon_1\right)+\varepsilon_1^2\left(\varepsilon_2-\gamma^2\right)\right)^2} + \frac{\varepsilon_3\sqrt{\left(\gamma^2-\varepsilon_3\right)^3}}{\left(\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)+\varepsilon_3^2\left(\varepsilon_2-\gamma^2\right)\right)^2},$$

$$k_2 = \frac{\varepsilon_1\sqrt{\gamma^2-\varepsilon_1}}{\varepsilon_2^2\left(\gamma^2-\varepsilon_1\right)+\varepsilon_1^2\left(\varepsilon_2-\gamma^2\right)} + \frac{\varepsilon_3\sqrt{\gamma^2-\varepsilon_3}}{\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)+\varepsilon_3^2\left(\varepsilon_2-\gamma^2\right)}.$$

Using (71)–(73) we write (50) at the point $\gamma=\gamma_0$, $a=0:$

$$\gamma_1 \equiv \left.\frac{d\gamma\,(a)}{da}\right|_{a=0} = \frac{\varepsilon_2^2\left(\gamma^2-\varepsilon_3\right)+\varepsilon_3^2\left(\varepsilon_2-\gamma^2\right)}{8\gamma\varepsilon_2^3\left(\varepsilon_2-\gamma^2\right)\left(\gamma^2-\varepsilon_3\right)}\frac{P_1\left(\gamma_0\right)}{Q_1\left(\gamma_0\right)}Z_h^2. \quad (74)$$

Now it is possible to find γ_1 using (74), and then expansion (49) is obtained.

Let us consider the function $F(a,\gamma)-h=0$ in the neighborhood of point $a=0$, $\gamma=\gamma_0$. The function $\tau=\tau(\eta)$ is a solution of algebraic equation (27) and coefficients of this algebraic equation are continuous functions with respect to a and γ. Taking this and formulas formulas (27), (29), and (34) into account we obtain that the function $F(a,\gamma)-h=0$ is continuous in the neighborhood of point $a=0$, $\gamma=\gamma_0$. As it is easy to see from formulas (51) and (52), in the neighborhood of point $a=0$, $\gamma=\gamma_0$ the function under consideration has partial derivatives with respect to a and γ. It follows from formula (73) that partial derivative with respect to γ does not vanish at the point $a=0$, $\gamma=\gamma_0$. Notice that $F(a,\gamma)-F(0,\gamma_0)=0$ at the point. This implies that the equation $F(a,\gamma)-h=0$ is unique solvable with respect to γ in a neighborhood of point $a=0$, $\gamma=\gamma_0$ and $\gamma\equiv\gamma(a)$. From formula (72) it is clear that partial derivative with respect to a of the function under consideration is also continuous at the point $a=0$, $\gamma=\gamma_0$. This implies that the function $\gamma\equiv\gamma(a)$ has a derivative at the point $a=0$ and this derivative is expressed by formula (50) [29]. This finishes the proof of validity of the first approximation. It should be noticed that all conclusions are made under conditions for ε_1, ε_2, ε_3, a, and γ (see §1).

CHAPTER 8

TM WAVE PROPAGATION IN AN ANISOTROPIC LAYER WITH KERR NONLINEARITY

§1. Statement of the Problem

Let us consider electromagnetic waves propagating through a homogeneous anisotropic nonmagnetic dielectric layer. The layer is located between two half-spaces: $x < 0$ and $x > h$ in Cartesian coordinate system $Oxyz$. The half-spaces are filled with isotropic nonmagnetic media without any sources and characterized by permittivities $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$, respectively, where ε_0 is the permittivity of free space[1]. Assume that everywhere $\mu = \mu_0$, where μ_0 is the permeability of free space.

The electromagnetic field depends on time harmonically [17]

$$\tilde{\mathbf{E}}(x,y,z,t) = \mathbf{E}_+(x,y,z)\cos\omega t + \mathbf{E}_-(x,y,z)\sin\omega t,$$
$$\tilde{\mathbf{H}}(x,y,z,t) = \mathbf{H}_+(x,y,z)\cos\omega t + \mathbf{H}_-(x,y,z)\sin\omega t,$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_+,\ \mathbf{E}_-,\ \tilde{\mathbf{H}},\ \mathbf{H}_+,\ \mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = (E_x, E_y, E_z)^T, \quad \mathbf{H} = (H_x, H_y, H_z)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

[1] Generally, conditions $\varepsilon_1 \geq \varepsilon_0$ and $\varepsilon_3 \geq \varepsilon_0$ are not necessary. They are not used for derivation of DEs, but they are useful for DEs' solvability analysis.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential field components on the media interfaces $x = 0$, $x = h$ and the radiation condition at infinity: the electromagnetic field exponentially decays as $|x| \to \infty$ in the domains $x < 0$ and $x > h$.

The permittivity inside the layer is described by the diagonal tensor

$$\tilde{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix},$$

where $\varepsilon_{xx} = \varepsilon_2 + b\,|E_x|^2 + a\,|E_z|^2$, $\varepsilon_{zz} = \varepsilon_2 + a\,|E_x|^2 + b\,|E_z|^2$; and a, b, $\varepsilon_2 > \max(\varepsilon_1, \varepsilon_3)$ are positive constants[1]. It does not matter what a form ε_{yy} has. Since ε_{yy} is not contained in the equations below for the TM case.

The solutions to the Maxwell equations are sought in the entire space.

The geometry of the problem is shown in Fig. 1.

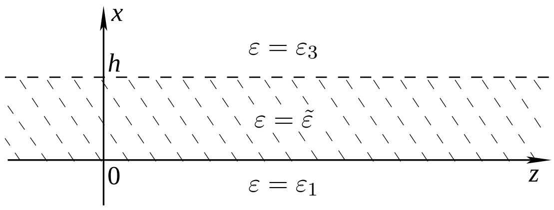


Fig. 1.

§2. TM Waves

Let us consider TM waves

$$\mathbf{E} = (E_x, 0, E_z)^T, \quad \mathbf{H} = (0, H_y, 0)^T,$$

where $E_x = E_x(x, y, z)$, $E_z = E_z(x, y, z)$, and $H_y = H_y(x, y, z)$.

[1] In §6 the solutions are sought under more general conditions.

Substituting the fields into Maxwell equations (1) we obtain

$$\begin{cases} \frac{\partial E_z}{\partial y} = 0, \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega\mu H_y, \\ \frac{\partial E_x}{\partial y} = 0, \\ \frac{\partial H_y}{\partial z} = i\omega\varepsilon_{xx} E_x, \\ \frac{\partial H_y}{\partial x} = -i\omega\varepsilon_{zz} E_z. \end{cases}$$

It is obvious from the first and the third equations of this system that $E_z = E_z(x, z)$ and $E_x = E_x(x, z)$ do not depend on y. It implies that H_y does not depend on y.

Waves propagating along medium interface z depend on z harmonically. This means that the fields components have the form

$$E_x = E_x(x)e^{i\gamma z},\ E_z = E_z(x)e^{i\gamma z},\ H_y = H_y(x)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system [17]

$$\begin{cases} i\gamma E_x(x) - E_z'(x) = i\omega\mu H_y(x), \\ H_y'(x) = -i\omega\varepsilon_{zz} E_z(x), \\ i\gamma H_y(x) = i\omega\varepsilon_{xx} E_x(x), \end{cases} \tag{2}$$

where $(\ \cdot\ )' \equiv \frac{d}{dx}$.

The following equation can be easily derived from the previous system

$$H_y(x) = \frac{1}{i\omega\mu}\left(i\gamma E_x(x) - E_z'(x)\right). \tag{3}$$

Differentiating equation (3) and using the second and the third equations of system (2) we obtain

$$\begin{cases} \gamma\left(iE_x\left(x\right)\right)' - E_z''\left(x\right) = \omega^2\mu\varepsilon_{zz}E_z\left(x\right), \\ \gamma^2\left(iE_x\left(x\right)\right) - \gamma E_z'\left(x\right) = \omega^2\mu\varepsilon_{xx}\left(iE_x\left(x\right)\right). \end{cases} \tag{4}$$

Let us denote by $k_0^2 := \omega^2\mu\varepsilon_0$ and perform the normalization according to the formulas $\tilde{x} = k_0 x$, $\frac{d}{dx} = k_0 \frac{d}{d\tilde{x}}$, $\tilde{\gamma} = \frac{\gamma}{k_0}$, $\tilde{\varepsilon}_j = \frac{\varepsilon_j}{\varepsilon_0}$ $(j = 1, 2, 3)$, $\tilde{a} = \frac{a}{\varepsilon_0}$, $\tilde{b} = \frac{b}{\varepsilon_0}$. Denoting by $Z(\tilde{x}) := E_z$, $X(\tilde{x}) := iE_x$ and omitting the tilde symbol, system (4) takes the form

$$\begin{cases} -Z'' + \gamma X' = \varepsilon_{zz} Z, \\ -Z' + \gamma X = \frac{1}{\gamma}\varepsilon_{xx} X. \end{cases} \tag{5}$$

It is necessary to find eigenvalues γ of the boundary eigenvalue problem that correspond to surface waves propagating along boundaries of the layer $0 < x < h$, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of spectral parameter γ such that real solutions $X(x)$ and $Z(x)$ to system (5) exist (see the footnote on p. 33 and also the remark on p. 74). We consider that

$$\varepsilon = \begin{cases} \varepsilon_1, & x < 0; \\ \tilde{\varepsilon}, & 0 < x < h; \\ \varepsilon_3, & x > h. \end{cases} \tag{6}$$

We assume that the spectral parameter γ satisfies the following two-sided inequality $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$. This inequality naturally occurs in the analogous problem for a linear layer (where the permittivity inside the layer is constant) (for further details see inequality (14) in Ch. 4).

Also we assume that functions X and Z are sufficiently smooth

$$\begin{aligned} X(x) \in & C\,(-\infty,\ 0] \cap C[0,h] \cap C\,[h,\ +\infty) \cap \\ & \cap C^1\,(-\infty,\ 0] \cap C^1[0,h] \cap C^1\,[h,\ +\infty)\,, \\ Z\,(x) \in & C(-\infty,+\infty) \cap C^1\,(-\infty,\ 0] \cap C^1[0,h] \cap C^1\,[h,\ +\infty) \cap \\ & \cap C^2(-\infty,0) \cap C^2(0,h) \cap C^2(h,+\infty). \end{aligned}$$

Physical nature of the problem implies these conditions.

It is clear that system (5) is an autonomous one. System (5) can be rewritten in a normal form (it will be done below). This system in the normal form can be considered as a dynamical system with analytical with respect to X and Z right-hand sides[1]. It is well known (see, for example [5]) that the solution X and Z of such a system are analytical functions with respect to independent variable as well. This is an important fact for DEs' derivation.

System (5) is the system for the anisotropic layer. Systems for the half-spaces can be easily obtained from system (5). For this purpose in system (5) it is necessary to put $\varepsilon_{xx} = \varepsilon_{zz} = \varepsilon$, where ε is the permittivity of the isotropic half-space.

We consider that γ satisfies the inequality $\gamma^2 > \max(\varepsilon_1, \varepsilon_3)$. This condition occurs in the case if at least one of the values ε_1 or ε_3 is positive. If both values ε_1 and ε_3 are negative, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem

In the domain $x < 0$ we have $\varepsilon = \varepsilon_1$. From system (5) we obtain the following system $X' = \gamma Z$, $Z' = \frac{\gamma^2 - \varepsilon_1}{\gamma} X$. From this system we obtain the equation $X'' = (\gamma^2 - \varepsilon_1)X$. Its general solution is $X(x) = A_1 e^{-x\sqrt{\gamma^2-\varepsilon_1}} + A e^{x\sqrt{\gamma^2-\varepsilon_1}}$. In accordance with the radiation condition we obtain the solution of the system

$$\begin{aligned} X(x) &= A \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right), \\ Z(x) &= \frac{\sqrt{\gamma^2-\varepsilon_1}}{\gamma} A \exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right). \end{aligned} \tag{7}$$

We assume that $\gamma^2 - \varepsilon_1 > 0$ otherwise it will be impossible to satisfy the radiation condition.

In the domain $x > h$ we have $\varepsilon = \varepsilon_3$. From system (5) we obtain the following system $X' = \gamma Z$, $Z' = \frac{\gamma^2 - \varepsilon_3}{\gamma} X$. From this system we obtain the equation $X'' = (\gamma^2 - \varepsilon_3)X$. Its general solution is $X(x) = Be^{-(x-h)\sqrt{\gamma^2-\varepsilon_3}} + B_1 e^{(x-h)\sqrt{\gamma^2-\varepsilon_3}}$. In accordance with the radiation condition we obtain the solution of the system

$$\begin{aligned} X(x) &= B \exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right), \\ Z(x) &= -\frac{\sqrt{\gamma^2-\varepsilon_3}}{\gamma} B \exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right). \end{aligned} \tag{8}$$

[1] Of course, in the domain where these right-hand sides are analytical with respect to X and Z.

Here for the same reason as above we consider that $\gamma^2 - \varepsilon_3 > 0$.

Constants A and B in (7) and (8) are defined by transmission conditions and initial conditions.

Inside the layer $0 < x < h$ system (5) takes the form

$$\begin{cases} -\frac{d^2 Z}{dx^2} + \gamma \frac{dX}{dx} = \left(\varepsilon_2 + aX^2 + bZ^2\right) Z, \\ -\frac{dZ}{dx} + \gamma X = \frac{1}{\gamma}\left(\varepsilon_2 + bX^2 + aZ^2\right) X. \end{cases} \tag{9}$$

System (9) can be rewritten in the following form[1]

$$\begin{cases} \frac{dX}{dx} = \frac{\gamma^2\left(\varepsilon_2 + aX^2 + bZ^2\right) + 2a\left(\varepsilon_2 + bX^2 + aZ^2 - \gamma^2\right)X^2}{\gamma\left(\varepsilon_2 + 3bX^2 + aZ^2\right)} Z, \\ \frac{dZ}{dx} = -\frac{1}{\gamma}\left(\varepsilon_2 - \gamma^2 + bX^2 + aZ^2\right) X. \end{cases} \tag{10}$$

From system (10) we obtain the ordinary differential equation

$$-\left(\varepsilon_2 + 3bX^2 + aZ^2\right)\frac{dX}{dZ} = \\ = 2aXZ + \gamma^2 \frac{\varepsilon_2 + aX^2 + bZ^2}{\varepsilon_2 + bX^2 + aZ^2 - \gamma^2}\frac{Z}{X}. \tag{11}$$

After multiplying by $\left(\varepsilon_2 + bX^2 + aZ^2 - \gamma^2\right) X$ equation (11) becomes a total differential equation. Its solution (the first integral of system (9)) can be easily found and can be written in the following form[2]

$$X^2\left(2\left(\varepsilon_2 + bX^2 + aZ^2\right)\left(\varepsilon_2 + bX^2 + aZ^2 - \gamma^2\right) - \gamma^2 bX^2\right) + \\ + \gamma^2\left(2\varepsilon_2 + bZ^2\right) Z^2 = C, \tag{12}$$

where C is a constant of integration.

[1]Now system (10) is written in a normal form. If the right-hand sides are analytic functions with respect to X and Z, then the solutions are analytic functions with respect to its independent variable. We notice it in the end of §2.

[2]All details about derivation see in Ch. 5 and Ch. 6.

§4. Transmission Conditions and the Transmission Problem

Tangential components of an electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are H_y and E_z. Hence we obtain

$$\begin{aligned} &H_y(h+0) = H_y(h-0), \quad H_y(0-0) = H_y(0+0), \\ &E_z(h+0) = E_z(h-0), \quad E_z(0-0) = E_z(0+0). \end{aligned}$$

From the continuity conditions for the tangential components of the fields $\mathbf{E}$ and $\mathbf{H}$ we obtain

$$\begin{aligned} &\gamma X(h) - Z'(h) = H_y^{(h)}, \qquad \gamma X(0) - Z'(0) = H_y^{(0)}, \\ &Z(h) = E_z(h+0) = E_z^{(h)}, \quad Z(0) = E_z(0-0) = E_z^{(0)}, \end{aligned} \tag{13}$$

where $H_y^{(h)} := i\frac{\sqrt{\mu}}{\sqrt{\varepsilon_0}}H_y(h+0)$, $H_y^{(0)} := i\frac{\sqrt{\mu}}{\sqrt{\varepsilon_0}}H_y(0-0)$.

The constant $E_z^{(h)} := E_z(h+0)$ is supposed to be known (the initial condition). Let us denote by $X_0 := X(0)$, $X_h := X(h)$, $Z_0 := Z(0)$, and $Z_h := Z(h)$. So we obtain that $A = \frac{\gamma}{\sqrt{\gamma^2-\varepsilon_3}}Z_0$, $B = \frac{\gamma}{\sqrt{\gamma^2-\varepsilon_3}}Z_h$.

Then from conditions (13) we obtain

$$H_y^{(h)} = -Z_h\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}, \quad H_y^{(0)} = Z_0\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}. \tag{14}$$

In accordance with (9) inside the layer

$$-Z'(x) + \gamma X(x) = \frac{1}{\gamma}\left(\varepsilon_2 + bX^2(x) + aZ^2(x)\right)X(x). \tag{15}$$

Then for $x = h$, using (13), we obtain from (15)

$$\frac{1}{\gamma}\left(\varepsilon_2 + bX_h^2 + aZ_h^2\right)X_h = H_y^{(h)}. \tag{16}$$

From (16) we obtain the equation with respect to X_h:

$$X_h^3 + \frac{\varepsilon_2 + aZ_h^2}{b}X_h - \frac{\gamma H_y^{(h)}}{b} = 0. \tag{17}$$

Under taken assumptions (in regard to ε_2, a, and b) the value $\frac{\varepsilon_2+aZ_h^2}{a} > 0$. Hence, equation (17) has at least one real root, which is considered

$$X_h = \left(\frac{\gamma H_y^{(h)}}{2b} + \sqrt{\frac{1}{27}\left(\frac{\varepsilon_2 + aZ_h^2}{b}\right)^3 + \frac{1}{4}\left(\frac{\gamma}{b}\right)^2 \left(H_y^{(h)}\right)^2} \right)^{1/3} + \\ + \left(\frac{\gamma H_y^{(h)}}{2b} - \sqrt{\frac{1}{27}\left(\frac{\varepsilon_2 + aZ_h^2}{b}\right)^3 + \frac{1}{4}\left(\frac{\gamma}{b}\right)^2 \left(H_y^{(h)}\right)^2} \right)^{1/3}.$$

Using first integral (12) at $x = h$, we find the value $C_h^X := C|_{x=h}$

$$C_h^X = \gamma^2 \left(2\varepsilon_2 + bZ_h^2\right) Z_h^2 - \gamma^2 b X_h^4 + \\ + 2X_h^2 \left(\varepsilon_2 + bX_h^2 + aZ_h^2\right) \left(\varepsilon_2 + bX_h^2 + aZ_h^2 - \gamma^2\right). \quad (18)$$

In order to find the values X_0 and Z_0 it is necessary to solve the following system[1]

$$\begin{cases} \frac{\gamma\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}} Z_0 = \left(\varepsilon_2 + bX_0^2 + aZ_0^2\right) X_0, \\ \gamma^2 \left(2\varepsilon_2 + bZ_0^2\right) Z_0^2 - \gamma^2 b X_0^4 + \\ \qquad + 2X_0^2 \left(\varepsilon_2 + bX_0^2 + aZ_0^2\right)\left(\varepsilon_2 + bX_0^2 + aZ_0^2 - \gamma^2\right) = C_h^X. \end{cases} \quad (19)$$

It is easy to see from the second equation of system (19) that the values X_0 and Z_0 can have arbitrary signs. At the same time from the first equation of this system we can see that X_0 and Z_0 have to be positive or negative simultaneously.

Normal components of electromagnetic field are known to be discontinues at media interfaces. And it is the discontinuity of the first kind. In this case the normal component is E_x. It is also known that the value εE_x is continuous at media interfaces. From the above

[1] This system is obtained using formula (15) at $x = 0$ and the first integral at the same point.

and from the continuity of the tangential component E_z it follows that the transmission conditions for the functions εX and Z are

$$[\varepsilon X]_{x=0} = 0, \quad [\varepsilon X]_{x=h} = 0, \quad [Z]_{x=0} = 0, \quad [Z]_{x=h} = 0, \qquad (20)$$

where $[f]_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$ denotes a jump of the function f at the interface.

We also suppose that functions $X(x)$ and $Z(x)$ satisfy the condition

$$X(x) = O\left(\frac{1}{|x|}\right) \quad \text{and} \quad Z(x) = O\left(\frac{1}{|x|}\right) \quad \text{as} \quad |x| \to \infty. \qquad (21)$$

Let

$$\mathrm{D} = \begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & \frac{d}{dx} \end{pmatrix}, \mathbf{F}(X, Z) = \begin{pmatrix} X \\ Z \end{pmatrix}, \mathbf{G}(\mathbf{F}, \gamma) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

where $X \equiv X(x)$, $Z \equiv Z(x)$ are unknown functions; $G_1 \equiv G_1(\mathbf{F}, \gamma)$, $G_2 \equiv G_2(\mathbf{F}, \gamma)$ are right-hand sides of system (10). The value γ is a spectral parameter. Rewrite the problem using new notations.

For the half-space $x < 0$ and $\varepsilon = \varepsilon_1$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & \gamma \\ \frac{\gamma^2 - \varepsilon_1}{\gamma} & 0 \end{pmatrix} \mathbf{F} = 0. \qquad (22)$$

Inside the layer $0 < x < h$ and $\varepsilon = \tilde{\varepsilon}$ we have

$$\mathrm{L}(\mathbf{F}, \gamma) \equiv \mathrm{D}\mathbf{F} - \mathbf{G}(\mathbf{F}, \gamma) = 0. \qquad (23)$$

For the half-space $x > h$ and $\varepsilon = \varepsilon_3$ we obtain

$$\mathrm{D}\mathbf{F} - \begin{pmatrix} 0 & \gamma \\ \frac{\gamma^2 - \varepsilon_3}{\gamma} & 0 \end{pmatrix} \mathbf{F} = 0. \qquad (24)$$

Let us formulate the transmission problem (it is possible to reformulate it as the boundary eigenvalue problem). *It is necessary to find eigenvalues* γ *and corresponding to them nonzero vectors* $\mathbf{F}$ *such that* $\mathbf{F}$ *satisfies to equations* (22)–(24). *Components* X, Z *of* *vector* **F** *satisfy transmission conditions* (20), *condition* (21) *and* X_0, Z_0 *satisfy to system* (19).

Definition 1. The value $\gamma = \gamma_0$ such that nonzero solution **F** to problem (22)–(24) exists under conditions (19)–(21) is called an eigenvalue of the problem. Solution **F** corresponding to the eigenvalue is called an eigenvector of the problem, and components $X(x)$ and $Z(x)$ of the vector **F** are called eigenfunctions (see the note on p. 37).

§5. Dispersion Equation

Introduce the new variables

$$\tau(x) = \frac{\varepsilon_2 + bX^2(x) + aZ^2(x)}{\gamma^2}, \quad \eta(x) = \gamma \frac{X(x)}{Z(x)} \tau(x). \tag{25}$$

Let $\tau_0 = \frac{\varepsilon_2}{\gamma^2}$, then

$$X^2 = \frac{\gamma^2 \eta^2 (\tau - \tau_0)}{b\eta^2 + a\gamma^2\tau^2}, \; Z^2 = \frac{\gamma^4 \tau^2 (\tau - \tau_0)}{b\eta^2 + a\gamma^2\tau^2}, \; XZ = \frac{\gamma^3 \tau \eta (\tau - \tau_0)}{b\eta^2 + a\gamma^2\tau^2}.$$

Using the new variables rewrite system (10) and equation (12)

$$\begin{cases} \frac{d\tau}{dx} = \frac{2\gamma^2\eta\tau^2(\tau-\tau_0)\left(\tau\left(b\eta^2+a\gamma^2\tau^2\right)(b-a(\tau-1))+b(a-b)(\tau-\tau_0)\left(\eta^2-\gamma^2\tau^2\right)\right)}{(b\eta^2+a\gamma^2\tau^2)(\tau(b\eta^2+a\gamma^2\tau^2)+2b(\tau-\tau_0)\eta^2)}, \\ \frac{d\eta}{dx} = \frac{\tau-1}{\tau}\eta^2 + \gamma^2\tau_0 + \gamma^2(\tau-\tau_0)\frac{a\eta^2+b\gamma^2\tau^2}{b\eta^2+a\gamma^2\tau^2}, \end{cases} \tag{26}$$

$$\eta^4 = \frac{2\gamma^2\tau^2}{b} \frac{(\tau - \tau_0)(a\tau(\tau-1) + b\tau_0) - a\left(C - \tau_0^2\right)}{C + 3\tau^2 - 2\tau^3 - 2\tau(2-\tau)\tau_0}\eta^2 + \\ + \frac{\gamma^4\tau^4}{b^2} \frac{b(\tau-\tau_0)(2a\tau_0 + b(\tau-\tau_0)) - a^2\left(C - \tau_0^2\right)}{C + 3\tau^2 - 2\tau^3 - 2\tau(2-\tau)\tau_0}, \tag{27}$$

the constant C is not equal to the value of the same name in (12).

Equation (27) is an algebraic sextic one with respect to τ and biquadratic one with respect to η.

In order to obtain the dispersion equation for propagation constants it is necessary to find the values $\eta(0)$, $\eta(h)$.

It is clear that $\eta(0) = \gamma\frac{X(0)}{Z(0)}\tau(0)$, $\eta(h) = \gamma\frac{X(h)}{Z(h)}\tau(h)$. Taking into account that $X(x)\tau(x) = \varepsilon X(x)$ and using formulas (13), (14), it is easy to obtain that

$$\eta(0) = \frac{\varepsilon_1}{\sqrt{\gamma^2 - \varepsilon_1}} > 0,\ \eta(h) = -\frac{\varepsilon_3}{\sqrt{\gamma^2 - \varepsilon_3}} < 0. \qquad (28)$$

The value C (we denote it as C_h^τ) can be easily find from first integral (27). Indeed, the values $\tau(h)$ and $\eta(h)$ are known, using first integral (27) at $x = h$ we obtain

$$C_h^\tau = \frac{1}{\frac{\varepsilon_3^4}{(\gamma^2-\varepsilon_3)^2} + \frac{2\gamma^2\tau^2}{b}\frac{\varepsilon_3^2}{\gamma^2-\varepsilon_3}a + \frac{\gamma^4\tau^4}{b^2}a^2}\times$$
$$\times\left(\frac{\varepsilon_3^4}{(\gamma^2-\varepsilon_3)^2}\left(-3\tau^2 + 4\tau_0\tau + 2\tau^2(\tau-\tau_0)\right) + \right.$$
$$+\frac{2\gamma^2\tau^2}{b}\frac{\varepsilon_3^2}{\gamma^2-\varepsilon_3}\left((\tau-\tau_0)\left(a\tau(\tau-1) + b\tau_0\right) + a\tau_0^2\right) +$$
$$\left. +\frac{\gamma^4\tau^4}{b^2}\left(b(\tau-\tau_0)\left(2a\tau_0 + b(\tau-\tau_0)\right) + a^2\tau_0^2\right)\right), \qquad (29)$$

where $\tau = \tau(h) = \frac{H_y^{(h)}}{\gamma X_h}$.

It is easy to see that the right-hand side of the second equation of system (26) is strictly positive. This means that the function $\eta(x)$ monotonically increases on the interval $(0, h)$. Taking into account (28) we obtain that the function $\eta(x)$ can not be differentiable on the entire interval $(0, h)$. This means that the function $\eta(x)$ has a break point. Let $x^* \in (0, h)$ be the break point. From (27) it is obvious that x^* is such that $\tau^* = \tau(x^*)$ is a root of the equation $C_h^\tau + 3(\tau^*)^2 - 2(\tau^*)^3 - 2\tau^*(2-\tau^*)\tau_0 = 0$. In addition $\eta(x^* - 0) \to +\infty$ and $\eta(x^* + 0) \to -\infty$.

It is natural to suppose that the function $\eta(x)$ on the interval $(0, h)$ has several break points $x_0,\ x_1,\ ...,\ x_N$. The properties of function $\eta(x)$ imply

$$\eta(x_i - 0) = +\infty,\ \eta(x_i + 0) = -\infty,\ \text{where } i = \overline{0, N}. \qquad (30)$$

Let

$$w := \frac{\tau}{(\tau-1)\eta^2 + \gamma^2\tau_0\tau + \gamma^2\tau(\tau-\tau_0)\frac{a\eta^2+b\gamma^2\tau^2}{b\eta^2+a\gamma^2\tau^2}},$$

where $w = w(\eta)$; $\tau = \tau(\eta)$ is expressed from equation (15).

Taking into account our hypothesis we will seek to the solutions on each interval $[0\ , x_0)$, (x_0, x_1), ..., $(x_N\ , h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0)} w d\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} w d\eta = x + c_i, & x_i \le x \le x_{i+1},\ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} w d\eta = x + c_N, & x_N \le x \le h. \end{cases} \tag{31}$$

Substituting $x = 0$, $x = x_{i+1}$, and $x = x_N$ into equation (31) (into the first, the second, and the third, respectively) and taking into account (30), we find constants $c_1, c_2, \ldots, c_{N+1}$:

$$\begin{cases} c_0 = -\int\limits_{\eta(0)}^{+\infty} w d\eta; \\ c_{i+1} = \int\limits_{-\infty}^{+\infty} w d\eta - x_{i+1}, \quad i = \overline{0, N-1}; \\ c_{N+1} = \int\limits_{-\infty}^{\eta(h)} w d\eta - h. \end{cases} \tag{32}$$

Using (32) we can rewrite equations (31) in the following form

$$\begin{cases} \int\limits_{\eta(x)}^{\eta(x_0)} w d\eta = -x + \int\limits_{\eta(0)}^{+\infty} w d\eta, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i)}^{\eta(x)} w d\eta = x + \int\limits_{-\infty}^{+\infty} w d\eta - x_{i+1}, & x_i \le x \le x_{i+1},\ i = \overline{0, N-1}; \\ \int\limits_{\eta(x_N)}^{\eta(x)} w d\eta = x + \int\limits_{-\infty}^{\eta(h)} w d\eta - h, & x_N \le x \le h. \end{cases} \tag{33}$$

Introduce the notation $T := \int\limits_{-\infty}^{+\infty} w d\eta$. It follows from formula (33) that $0 < x_{i+1} - x_i = T < h$, where $i = \overline{0, N-1}$. This implies the convergence of the improper integral (it will be proved in other way below). Now consider x in equations (33) such that all the integrals on the left side vanish (i.e. $x = x_0$, $x = x_i$, and $x = x_N$), and sum all equations (33). We obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{+\infty} w d\eta + x_0 + T - x_1 + ... + x_{N-1} + T - x_N + x_N + \int\limits_{-\infty}^{\eta(h)} w d\eta - h.$$

This formula implies

$$-\int\limits_{-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}}^{\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}} w d\eta + (N+1)\, T = h. \tag{34}$$

Expression (34) is the DE, which holds for any finite h. Let γ be a solution of DE (34) and an eigenvalue of the problem. Then, there are eigenfunctions X and Z, which correspond to the eigenvalue γ. The eigenfunction Z has $N+1$ zeros on the interval $(0, h)$.

Notice that improper integrals in DE (34) converge. Indeed, function $\tau = \tau(\eta)$ is bounded as $\eta \to \infty$ since $\tau = \frac{\varepsilon_2 + bX^2 + aZ^2}{\gamma^2}$, and X, Z are bounded. Then

$$|w| = \left| \frac{\tau}{(\tau - 1)\eta^2 + \gamma^2 \tau_0 \tau + \gamma^2 \tau (\tau - \tau_0) \frac{a\eta^2 + b\gamma^2\tau^2}{b\eta^2 + a\gamma^2\tau^2}} \right| \leq \frac{1}{\alpha \eta^2 + \beta},$$

where $\alpha > 0$, $\beta > 0$ are constants. It is obvious that improper integral $\int\limits_{-\infty}^{\infty} \frac{d\eta}{\alpha\eta^2 + \beta}$ converges. Convergence of the improper integrals in (34) in inner points results from the requirement that the right-hand side of the second equation of system (26) is positive.

The first equation of system (26) jointly with the first integral can be integrated in Abelian functions. The solution is expressed in implicit form by means of Abelian integrals. The inversion of these integrals are Abelian functions and they are solutions of system (26). Abelian functions are meromorphic and periodic ones. Since function η is expressed algebraically through τ; therefore η is a meromorphic periodic function. This means that the break point x^* is a pole of function η [6, 36].

Theorem 1 (*of equivalence*). *Boundary eigenvalue problem* (22)–(24) *with conditions* (19)–(21) *has a solution* (*an eigenvalue*) *if and only if this eigenvalue is a solution of DE* (34).

Proof. *Sufficiency.* It is obvious that if we find the solution γ of DE (34), then we can find functions $\tau(x)$ and $\eta(x)$ from system (26) and first integral (27). From functions $\tau(x)$ and $\eta(x)$, and using formulas (25) we find

$$X(x) = \pm\gamma\eta\sqrt{\frac{\tau-\tau_0}{b\eta^2+a\gamma^2\tau^2}} \text{ and } Z(x) = \pm\gamma^2\tau\sqrt{\frac{\tau-\tau_0}{b\eta^2+a\gamma^2\tau^2}}. \tag{35}$$

It is an important question how to choose the signs. Let us discuss it in detail. We know the behavior of the function $\eta = \gamma\tau\frac{X}{Z}$: it monotonically increases, and if $x = x^*$ such that $\eta(x^*) = 0$, then $\eta(x^*-0) < 0$, $\eta(x^*+0) > 0$, and if $x = x^{**}$ such that $\eta(x^{**}) = \pm\infty$, then $\eta(x^{**}-0) > 0$ and $\eta(x^{**}+0) < 0$. Function η has no other points of sign's reversal. To fix the idea, assume that the initial condition is $Z(h) > 0$. If $\eta > 0$, then functions X and Z have the same signs; if $\eta < 0$, the functions X and Z have different signs. Since X and Z are continuous[1] we can choose necessary signs in expressions (35).

Necessity. It follows from the method of obtaining of DE (34) from system (26) that an eigenvalue of the problem is a solution of the DE.

[1]Of course, we mean that X and Z are continuous functions in the domains $x < 0$, $0 < x < h$, and $x > h$.

It should be also noticed that eigenfunctions (or eigenmodes) that correspond an eigenvalue γ_0 can be easily numerically calculated from system (9) or (10), (for example, using a Runge-Kutta method).

Introduce the notation $J(\gamma,k) := \int\limits_{\eta(0)}^{\eta(h)} w d\eta + kT$, where the right-hand side is defined by DE (34) and $k = \overline{0, N+1}$.

Let

$$h_{\inf}^k = \inf_{\gamma^2 \in (\max(\varepsilon_1,\varepsilon_3),\varepsilon_2)} J(\gamma,k),$$
$$h_{\sup}^k = \sup_{\gamma^2 \in (\max(\varepsilon_1,\varepsilon_3),\varepsilon_2)} J(\gamma,k).$$

Let us formulate the sufficient condition of existence at least one eigenvalue of the theorem.

Theorem 2. *Let h satisfies for a certain $k = \overline{0, N+1}$ the following two-sided inequality*

$$h_{\inf}^k < h < h_{\sup}^k,$$

then boundary eigenvalue problem (22)–(24) *with conditions* (19)–(21) *has at least oe solution* (*an eigenvalue*).

The quantities $h_{\inf}^k$ and $h_{\sup}^k$ can be numerically calculated.

§6. Generalized Dispersion Equation

Here we derive the generalized DE, which holds for any real values ε_2. In addition the sign of the right-hand side of the second equation in system (26) (see the footnote on p. 43), and conditions $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$ or $0 < \gamma^2 < \varepsilon_2$ are not taken into account. These conditions appear in the case of a linear layer and are used for derivation of DE (34). Though on the nonlinear case it is not necessary to limit the value γ^2 from the right side. At the same time it is clear that γ is limited from the left side, since this limit appears from the solutions in the half-spaces (where the permittivities are constants).

Now we assume that γ satisfies one of the following inequalities

$$\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < +\infty,$$

when at least one of the values ε_1 or ε_3 is positive, or

$$0 < \gamma^2 < +\infty,$$

when both $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$.

At first we derive the DE from system (26) and first integral (27), and then we discuss the details of derivation and conditions when the derivation is possible and the DE is well defined.

Using first integral (27) it is possible to integrate formally any of the equations of system (26). As earlier we shall integrate the second equation. We can not obtain the solution on the entire interval $(0, h)$, since function $\eta(x)$ can have break points, which belong to $(0, h)$. It is known that function $\eta(x)$ has break points only of the second kind (η is an analytical function).

Assume that function $\eta(x)$ on interval $(0, h)$ has $N + 1$ break points x_0, x_1, ..., x_N.

It should be noticed that

$$\eta(x_i - 0) = \pm\infty, \quad \eta(x_i + 0) = \pm\infty,$$

where $i = \overline{0,\ N}$, and signs $\pm$ are independent and unknown.

Taking into account the above, solutions are sought on each interval $[0\ , x_0)$, (x_0, x_1), ..., $(x_N\ , h]$:

$$\begin{cases} -\int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = x + c_0, & 0 \le x \le x_0; \\ \int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + c_{i+1}, & x_i \le x \le x_{i+1},\ i = \overline{0,\ N-1}; \\ \int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + c_{N+1}, & x_N \le x \le h. \end{cases} \quad (36)$$

From equations (36), substituting $x = 0$, $x = x_{i+1}$, and $x = x_N$ into the first, the second, and the third equations (36), respectively, we find required constants c_1, c_2, ..., c_{N+1}:

$$
\begin{cases}
c_0 = -\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta; \\
c_{i+1} = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, \quad i = \overline{0, N-1}; \\
c_{N+1} = \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h.
\end{cases}
\tag{37}
$$

Using (37) equations (36) take the form

$$
\begin{cases}
\int\limits_{\eta(x)}^{\eta(x_0-0)} w d\eta = -x + \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta, & 0 \le x \le x_0; \\
\int\limits_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, & x_i \le x \le x_{i+1}; \\
\int\limits_{\eta(x_N+0)}^{\eta(x)} w d\eta = x + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h, & x_N \le x \le h,
\end{cases}
\tag{38}
$$

where $i = \overline{0, N-1}$.

From formulas (38) we obtain that

$$
x_{i+1} - x_i = \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta, \tag{39}
$$

where $i = \overline{0, N-1}$.

Expressions $0 < x_{i+1} - x_i < h < \infty$ imply that under the assumption about the break points existence the integral on the right side converges and $\int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta > 0$. In the same way, from the first and the last equations (38) we obtain that $x_0 = \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta$ and $0 < x_0 < h$ then

$$
0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta < h < \infty;
$$

and $h - x_N = \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta$ and $0 < h - x_N < h$ then

$$0 < \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta < h < \infty.$$

These considerations yield that the function $w(\eta)$ has no non-integrable singularities for $\eta \in (-\infty, \infty)$. And also this proves that the assumption about a finite number break points is true.

Now, setting $x = x_0$, $x = x_i$, and $x = x_N$ into the first, the second, and the third equations in (38), respectively, we have that all the integrals on the left sides vanish. We add all the equations in (38) to obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + x_0 + \int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} w d\eta - x_1 + ...$$

$$... + x_{N-1} + \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} w d\eta - x_N + x_N + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta - h. \quad (40)$$

From (40) we obtain

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} w d\eta + \sum_{i=0}^{N-1} \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta = h. \quad (41)$$

It follows from formulas (39) that

$$\eta\,(x_i + 0) = \pm\infty \text{ and } \eta\,(x_i - 0) = \mp\infty, \text{ where } i = \overline{0, N},$$

and it is necessary to choose the infinities of different signs.

Thus we obtain

$$\int\limits_{\eta(x_0+0)}^{\eta(x_1-0)} w d\eta = ... = \int\limits_{\eta(x_{N-1}+0)}^{\eta(x_N-0)} w d\eta =: T'.$$

Hence $x_1 - x_0 = ... = x_N - x_{N-1}$.

Now we can rewrite equation (41) in the following form

$$\int\limits_{\eta(0)}^{\eta(x_0-0)} w d\eta + \int\limits_{\eta(x_N+0)}^{\eta(h)} f d\eta + NT' = h.$$

Let $T := \int\limits_{-\infty}^{+\infty} w d\eta$, then we finally obtain

$$-\int\limits_{\eta(h)}^{\eta(0)} w d\eta \pm (N+1)T = h, \tag{42}$$

where $\eta(0)$, $\eta(h)$ are defined by formulas (28).

Expression (42) is the DE, which holds for any finite h. Let γ be a solution of DE (42) and an eigenvalue of the problem. Then, there are eigenfunctions X and Z, which correspond to the eigenvalue γ. The eigenfunction Z has $N+1$ zeros on the interval $(0, h)$. It should be noticed that for every number $N+1$ it is necessary to solve two DEs: for $N+1$ and for $-(N+1)$.

Let us formulate the following

Theorem 3. *The set of solutions of DE* (42) *contains the set of solutions* (*eigenvalues*) *of boundary eigenvalue problem* (22)–(24) *with conditions* (19)–(21).

Proof. It is obvious that this theorem generalises Theorem 1. It is also obvious that any eigenvalue of the problem is a solution of the DE. It is easy to understand where additional solutions of the DE occur from (the solutions, which are not eigenvalues). If the values ε_2, a, and b are arbitrary real values, then equation (17) and system (19) can have several roots. And it is possible that among these roots we can not choose roots that correspond to the problem. In other words, for each group of three roots we have DE (42). It is clear that not all the solutions of these DE are eigenvalues of the problem. A solution of the DE is an eigenvalue of the problem if and only if transmission conditions (20) are satisfied. That is, if we have a solution γ of the DE, on the one hand, then we can find X_0, Z_0, and X_h. On the other hand, we can find the values X_0, Z_0, and X_h from equation (17) and system (19). The solution γ is an eigenvalue if and only if each value found in one way coincides with corresponding value found in other way. Using this criterion we can determine eigenvalues among solutions of the DE. This criterion can be easily used for numerical calculation.

Now, let us review some theoretical treatments of derivation of DEs (34) and (42). We are going to discuss the existence and uniqueness of system's (10) solutions.

Let us consider vector form (23) of system (10):

$$\mathrm{D}\mathbf{F} = \mathbf{G}(\mathbf{F}, \lambda). \tag{43}$$

Let the right-hand side $\mathbf{G}$ be defined and continuous in the domain $\Omega \subset \mathbb{R}^2$, $\mathbf{G} : \Omega \to \mathbb{R}^2$. Also we suppose that $\mathbf{G}$ satisfies the Lipschitz condition on $\mathbf{F}$ (locally in Ω)[1].

Under these conditions system (10) (or (43)) has a unique solution in the domain Ω [8, 41, 22].

It is clear that under these conditions system (26) has a unique solution (of course, the domain of uniqueness Ω' for variables τ, η differs from Ω).

Since we seek bounded solutions X and Z; therefore, we obtain

$$\Omega \subset [-m_1, m_1] \times [-m_2, m_2],$$

where

$$\max_{x\in[0,h]} |Y| < m_1, \quad \max_{x\in[0,h]} |Z| < m_2$$

and the previous implies that

$$\Omega' \subset [\varepsilon_f, \varepsilon_f + m_1^2] \times (-\infty, +\infty).$$

It is easy to show that there is no point $x^* \in \Omega'$, such that $X|_{x=x^*} = 0$ and $Z|_{x=x^*} = 0$. Indeed, it is known from theory of autonomous system (see, for example, [41]) that phase trajectories

[1]About the Lipschitz condition see the footnote on p. 48.

do not intersect one another in the system's phase space when right-hand side of the system is continuous and satisfies the Lipschitz condition. Since $\tilde{X} \equiv 0$ and $\tilde{Z} \equiv 0$ are stationary solutions of system (10), it is obvious that the nonconstant solutions X and Z can not vanish simultaneously at a certain point $x^* \in \Omega'$ (otherwise the nonconstant solutions intersect with the stationary solutions and we obtain a contradiction).

Note 1. If there is a certain value γ_*^2 such that some of the integrals in DEs (34) or (42) diverge at certain inner points, then this simply means that the value γ_*^2 is not a solution of chosen DE and the value γ_*^2 is not an eigenvalue of the problem.

Note 2. This problem depends on the initial condition, see the note on p. 49 for further details.

We derived the DEs from the second equation of system (26). It is possible to do it using the first equation of the system (see p. 130).

For positive values of parameters a, b, and ε_2 the behavior of dispersion curves are shown in Fig. 2, 3 Ch. 7 (see p. 132, 132).

PART II

BOUNDARY EIGENVALUE PROBLEMS
FOR THE MAXWELL EQUATIONS
IN CIRCLE CYLINDRICAL WAVEQUIDES

CHAPTER 9

TE AND TM WAVES GUIDED BY A CIRCLE CYLINDRICAL WAVEGUIDE

Let us consider three-dimensional space $\mathbb{R}^3$ with Cartesian coordinate system $Oxyz$. The space is filled by isotropic medium with constant permittivity $\varepsilon \geq \varepsilon_0$, where ε_0 is the permittivity of free space. In this medium a circle cylindrical waveguide is placed. The waveguide is filled by anisotropic nonmagnetic medium. The waveguide has cross section $W := \left\{(x,y) : x^2 + y^2 < R^2\right\}$ and its generating line (the waveguide axis) is parallel to the axis Oz. We shall consider electromagnetic waves propagating along the waveguide axis, that is eigenmodes of the structure.

Let us consider also cylindrical coordinate system $O\rho\varphi z$. Axis Oz of cylindrical coordinate system coincides with axis Oz of Cartesian coordinate system.

The electromagnetic field depends on time harmonically [17]

$$\begin{aligned}\tilde{\mathbf{E}}\left(\rho,\ \varphi,\ z,\ t\right) &= \mathbf{E}_+\left(\rho,\ \varphi,\ z\right)\cos\omega t + \mathbf{E}_-\left(\rho,\ \varphi,\ z\right)\sin\omega t,\\ \tilde{\mathbf{H}}\left(\rho,\ \varphi,\ z,\ t\right) &= \mathbf{H}_+\left(\rho,\ \varphi,\ z\right)\cos\omega t + \mathbf{H}_-\left(\rho,\ \varphi,\ z\right)\sin\omega t,\end{aligned}$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_+,\ \mathbf{E}_-,\ \tilde{\mathbf{H}},\ \mathbf{H}_+,\ \mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_+ + i\mathbf{E}_-,\\ \mathbf{H} &= \mathbf{H}_+ + i\mathbf{H}_-,\end{aligned}$$

where

$$\mathbf{E} = \left(E_\rho, E_\varphi, E_z\right)^T, \quad \mathbf{H} = \left(H_\rho, H_\varphi, H_z\right)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and

$$\begin{aligned}&E_\rho = E_\rho(\rho,\ \varphi, z), \quad E_\varphi = E_\varphi(\rho,\ \varphi, z), \quad E_z = E_z(\rho,\ \varphi, z),\\ &H_\rho = H_\rho(\rho,\ \varphi, z), \quad H_\varphi = H_\varphi(\rho,\ \varphi, z), \quad H_z = H_z(\rho,\ \varphi, z).\end{aligned}$$

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential components on the media interfaces (on the boundary of the waveguide) and the radiation condition at infinity: the electromagnetic field exponentially decays as $\rho \to \infty$.

The permittivity in the waveguide is described by the diagonal tensor

$$\tilde{\varepsilon} = \begin{pmatrix} \varepsilon_{\rho\rho} & 0 & 0 \\ 0 & \varepsilon_{\varphi\varphi} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix},$$

where $\varepsilon_{\rho\rho}$, $\varepsilon_{\varphi\varphi}$, ε_{zz} are constants.

The solutions to the Maxwell equations are sought in the entire space.

The geometry of the problem is shown on Fig. 1. The waveguide is infinite along axis Oz.

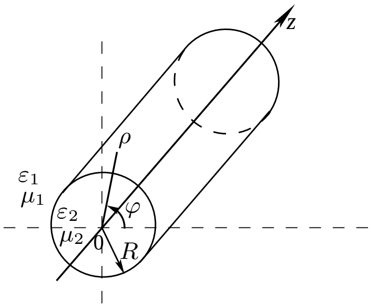


Fig. 1.

Rewrite system (1) in the expanded form

$$\begin{cases} \frac{1}{\rho}\frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} = -i\omega\varepsilon_{\rho\rho}E_\rho, \\ \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -i\omega\varepsilon_{\varphi\varphi}E_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho H_\varphi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \varphi} = -i\omega\varepsilon_{zz}E_z, \end{cases} \qquad \begin{cases} \frac{1}{\rho}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} = i\omega\mu H_\rho, \\ \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = i\omega\mu H_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho E_\varphi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial E_\rho}{\partial \varphi} = i\omega\mu H_z. \end{cases} \tag{2}$$

Since the waveguide structure has circular symmetry; therefore, sought-for solutions are periodical functions with respect to φ. This means that the components of the electromagnetic field have the form

$$\begin{aligned} &E_\rho = E_\rho(\rho, z)e^{in\varphi}, \quad E_\varphi = E_\varphi(\rho, z)e^{in\varphi}, \quad E_z = E_z(\rho, z)e^{in\varphi}, \\ &H_\rho = H_\rho(\rho, z)e^{in\varphi}, \quad H_\varphi = H_\varphi(\rho, z)e^{in\varphi}, \quad H_z = H_z(\rho, z)e^{in\varphi}. \end{aligned} \tag{3}$$

where $n = 0, 1, 2, \ldots$

Taking into account formulas (3) system (2) takes the form

$$\begin{cases} \frac{in}{\rho}H_z - \frac{\partial H_\varphi}{\partial z} = -i\omega\varepsilon_{\rho\rho}E_\rho, \\ \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -i\omega\varepsilon_{\varphi\varphi}E_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho H_\varphi)}{\partial \rho} - \frac{in}{\rho}H_\rho = -i\omega\varepsilon_{zz}E_z, \end{cases} \qquad \begin{cases} \frac{in}{\rho}E_z - \frac{\partial E_\varphi}{\partial z} = i\omega\mu H_\rho, \\ \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = i\omega\mu H_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho E_\varphi)}{\partial \rho} - \frac{in}{\rho}E_\rho = i\omega\mu H_z. \end{cases} \tag{4}$$

Setting $n = 0$, from (4) we obtain

$$\begin{cases} \frac{\partial H_\varphi}{\partial z} = i\omega\varepsilon_{\rho\rho}E_\rho, \\ \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = i\omega\mu H_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho H_\varphi)}{\partial \rho} = -i\omega\varepsilon_{zz}E_z, \end{cases} \qquad \begin{cases} \frac{\partial E_\varphi}{\partial z} = -i\omega\mu H_\rho, \\ \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -i\omega\varepsilon_{\varphi\varphi}E_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho E_\varphi)}{\partial \rho} = i\omega\mu H_z. \end{cases} \tag{5}$$

Thus system (1) breaks up into two independent systems (5). It follows from system (5) that it is possible to consider two (independent) polarizations of electromagnetic field $(\mathbf{E}, \mathbf{H})$. The one corresponds to the first system in (5) and has the form

$$\mathbf{E} = (E_\rho, 0, E_z)^T, \quad \mathbf{H} = (0, H_\varphi, 0)^T. \tag{6}$$

And the other one corresponds to the second system in (5) and has the form

$$\mathbf{E} = (0, E_\varphi, 0)^T, \quad \mathbf{H} = (H_\rho, 0, H_z)^T. \tag{7}$$

Waves (solutions to the Maxwell equations) propagating along waveguide axis depend on z harmonically.

From the above we obtain for waves (6)

$$E_\rho = E_\rho(\rho)e^{i(n\varphi+\gamma z)}, \;\; E_z = E_z(\rho)e^{i(n\varphi+\gamma z)}, \;\; H_\varphi = H_\varphi(\rho)e^{i(n\varphi+\gamma z)},$$

where γ is the propagation constant.

Substituting these components into the first system in (5) we obtain

$$\begin{cases} i\gamma H_\varphi = i\omega\varepsilon_{\rho\rho}E_\rho, \\ i\gamma E_\rho - \frac{dE_z}{d\rho} = i\omega\mu H_\varphi, \\ \frac{1}{\rho}\frac{d(\rho H_\varphi)}{d\rho} = -i\omega\varepsilon_{zz}E_z, \end{cases}$$

where functions E_ρ, E_z, H_φ depend on variable ρ. Hence we have the system of ordinary differential equations.

From the above we obtain for waves (7)

$$E_\varphi = E_\varphi(\rho)e^{i(n\varphi+\gamma z)}, \;\; H_\rho = H_\rho(\rho)e^{i(n\varphi+\gamma z)}, \;\; H_z = H_z(\rho)e^{i(n\varphi+\gamma z)},$$

where γ is the spectral parameter (propagation constant).

Substituting these components into the second system in (5) we obtain

$$\begin{cases} i\gamma E_\varphi = -i\omega\mu H_\rho, \\ i\gamma H_\rho - \frac{dH_z}{d\rho} = -i\omega\varepsilon_{\varphi\varphi}E_\varphi, \\ \frac{1}{\rho}\frac{d(\rho E_\varphi)}{d\rho} = i\omega\mu H_z, \end{cases}$$

where functions E_φ, H_ρ, H_z depend on variable ρ. Hence we have the system of ordinary differential equations again.

In both cases (6) and (7) it is possible to consider each component of electromagnetic field as a function of three spatial variables (ρ, φ, z). After substituting these components into system (1) we obtain that each component does not depend on φ.

Waves (6) are called TM-polarized electromagnetic waves[1], or simply TM waves.

Waves (7) are called TE-polarized electromagnetic waves[2], or simply TE waves.

---

[1] transverse-magnetic.

[2] transverse-electric.

The discussion about TE and TM waves in linear and nonlinear media see on the p. 19, with the replacement of a layer by a cylindrical waveguide.

In the case when $n \neq 0$, there are no such simple waves as (6) and (7) (see, for example, [19]). But also two polarizations (TE and TM waves, respectively) exist

$$\mathbf{E} = (E_\rho, E_\varphi, 0)^T, \ \mathbf{H} = (H_\rho, H_\varphi, H_z)^T \tag{8}$$

and

$$\mathbf{E} = (E_\rho, E_\varphi, E_z)^T, \ \mathbf{H} = (H_\rho, H_\varphi, 0)^T. \tag{9}$$

Let us review the consequences of using polarizations (8) and (9) for linear and nonlinear media inside the waveguide.

Let us consider polarized waves (8), where the components have the form

$$E_\rho = E_\rho(\rho)e^{i(n\varphi+\gamma z)}, \ \ E_\varphi = E_\varphi(\rho)e^{i(n\varphi+\gamma z)}, \ E_z = 0,$$
$$H_\rho = H_\rho(\rho)e^{i(n\varphi+\gamma z)}, \ H_\varphi = H_\varphi(\rho)e^{i(n\varphi+\gamma z)}, \ H_z = H_z(\rho)e^{i(n\varphi+\gamma z)}.$$

System (4) takes the form

$$\begin{cases} \frac{in}{\rho}H_z - i\gamma H_\varphi = -i\omega\varepsilon_{\rho\rho}E_\rho, \\ i\gamma H_\rho - \frac{\partial H_z}{\partial \rho} = -i\omega\varepsilon_{\varphi\varphi}E_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho H_\varphi)}{\partial \rho} - \frac{in}{\rho}H_\rho = 0, \end{cases} \qquad \begin{cases} -i\gamma E_\varphi = i\omega\mu H_\rho, \\ i\gamma E_\rho = i\omega\mu H_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho E_\varphi)}{\partial \rho} - \frac{in}{\rho}E_\rho = i\omega\mu H_z. \end{cases} \tag{10}$$

From the forth and the fifth equations of system (10) we obtain

$$E_\varphi = -\frac{\omega\mu}{\gamma}H_\rho \quad \text{and} \quad E_\rho = \frac{\omega\mu}{\gamma}H_\varphi. \tag{11}$$

Taking (11) into account from the first and the second equations of system (10) we find

$$H_z = \frac{1}{n}\left(\gamma - \frac{\omega^2\mu\varepsilon_{\rho\rho}}{\gamma}\right)\rho H_\varphi \quad \text{and} \quad \frac{\partial H_z}{\partial \rho} = i\left(\gamma - \frac{\omega^2\mu\varepsilon_{\varphi\varphi}}{\gamma}\right)H_\rho. \tag{12}$$

On the other hand, from the third equation of system (10) we obtain

$$H_\rho = \frac{1}{in}\frac{\partial(\rho H_\varphi)}{\partial\rho}. \tag{13}$$

From formulas (12) and (13) we obtain

$$\frac{\omega^2\mu}{\gamma}(\varepsilon_{\rho\rho} - \varepsilon_{\varphi\varphi})\frac{\partial(\rho H_\varphi)}{\partial\rho} = \rho H_\varphi\frac{\partial}{\partial\rho}\left(\gamma - \frac{\omega^2\mu\varepsilon\rho\rho}{\gamma}\right). \tag{14}$$

If $\varepsilon_{\rho\rho}$ does not depend on ρ, then equation (14) implies

$$\frac{\omega^2\mu}{\gamma}(\varepsilon_{\rho\rho} - \varepsilon_{\varphi\varphi})\frac{\partial(\rho H_\varphi)}{\partial\rho} = 0. \tag{15}$$

It is easy to show that equation (15) implies the necessary inference that

$$\varepsilon_{\rho\rho} = \varepsilon_{\varphi\varphi}.$$

This means that for a constant permittivity tensor and for polarization (8) the equality $\varepsilon_{\rho\rho} = \varepsilon_{\varphi\varphi}$ is always fulfilled. In other words, for such a field the waveguide is always isotropic along axes ρ and φ.

If $\varepsilon_{\rho\rho} = \varepsilon_{\varphi\varphi} \neq const$ are functions with respect to $|\mathbf{E}|$ (the nonlinear case!), then equation (14) implies

$$\text{either}\ \ \rho H_\varphi = 0, \ \ \text{or}\ \ \frac{\partial}{\partial\rho}\left(\gamma - \frac{\omega^2\mu\varepsilon_{\rho\rho}}{\gamma}\right) = 0.$$

The case $\rho H_\varphi = 0$ implies that each component of the electromagnetic field vanishes.

The case $\frac{\partial}{\partial\rho}\left(\gamma - \frac{\omega^2\mu\varepsilon_{\rho\rho}}{\gamma}\right) = 0$ implies that $\varepsilon_{\rho\rho}$ does not depend on ρ. This contradicts the initial assumption.

All this means that in a nonlinear waveguide (isotropic along axes ρ, φ) there are no nonlinear eigenmodes with polarization (8)! It should be noticed that in a linear waveguide there are eigenmodes for polarization (8).

Now, let us consider polarized waves (9), where the components have the form

$$E_\rho = E_\rho(\rho)e^{i(n\varphi+\gamma z)},\ \ E_\varphi = E_\varphi(\rho)e^{i(n\varphi+\gamma z)},\ \ E_z = E_z(\rho)e^{i(n\varphi+\gamma z)},$$
$$H_\rho = H_\rho(\rho)e^{i(n\varphi+\gamma z)},\ H_\varphi = H_\varphi(\rho)e^{i(n\varphi+\gamma z)},\ H_z = 0.$$

System (4) takes the form

$$\begin{cases} -i\gamma H_\varphi = -i\omega\varepsilon_{\rho\rho}E_\rho, \\ i\gamma H_\rho = -i\omega\varepsilon_{\varphi\varphi}E_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho H_\varphi)}{\partial\rho} - \frac{in}{\rho}H_\rho = -i\omega\varepsilon_{zz}E_z, \end{cases} \qquad \begin{cases} \frac{in}{\rho}E_z - i\gamma E_\varphi = i\omega\mu H_\rho, \\ i\gamma E_\rho - \frac{\partial E_z}{\partial\rho} = i\omega\mu H_\varphi, \\ \frac{1}{\rho}\frac{\partial(\rho E_\varphi)}{\partial\rho} - \frac{in}{\rho}E_\rho = 0. \end{cases} \tag{16}$$

From the first and the second equations of system (10) we obtain

$$H_\varphi = \frac{\omega\varepsilon_{\rho\rho}}{\gamma}E_\rho \quad \text{and} \quad H_\rho = -\frac{\omega\varepsilon_{\varphi\varphi}}{\gamma}E_\varphi. \tag{17}$$

Taking (17) into account from the forth and the fifth equations of system (16) we find

$$E_z = \frac{1}{n}\left(\gamma - \frac{\omega^2\mu\varepsilon_{\varphi\varphi}}{\gamma}\right)\rho E_\varphi \quad \text{and} \quad \frac{\partial E_z}{\partial\rho} = i\left(\gamma - \frac{\omega^2\mu\varepsilon_{\rho\rho}}{\gamma}\right)E_\rho. \tag{18}$$

On the other hand, from the sixth equation of system (16) we obtain

$$E_\rho = \frac{1}{in}\frac{\partial(\rho E_\varphi)}{\partial\rho}. \tag{19}$$

From formulas (18) and (19) we obtain

$$\frac{\omega^2\mu}{\gamma}(\varepsilon_{\varphi\varphi} - \varepsilon_{\rho\rho})\frac{\partial(\rho E_\varphi)}{\partial\rho} = \rho E_\varphi\frac{\partial}{\partial\rho}\left(\gamma - \frac{\omega^2\mu\varepsilon_{\varphi\varphi}}{\gamma}\right). \tag{20}$$

If $\varepsilon_{\varphi\varphi}$ does not depend on ρ, then equation (20) implies

$$\frac{\omega^2\mu}{\gamma}(\varepsilon_{\varphi\varphi} - \varepsilon_{\rho\rho})\frac{\partial(\rho H_\varphi)}{\partial\rho} = 0. \tag{21}$$

It is easy to show that equation (21) implies the necessary inference that

$$\varepsilon_{\rho\rho} = \varepsilon_{\varphi\varphi}.$$

This means that for a constant permittivity tensor and for polarization (9) the equality $\varepsilon_{\rho\rho} = \varepsilon_{\varphi\varphi}$ is always executed. In other words, for such a field the waveguide is always isotropic along axes ρ and φ.

If $\varepsilon_{\rho\rho} = \varepsilon_{\varphi\varphi} \neq const$ are functions with respect to $|\mathbf{E}|$ (the nonlinear case!), then equation (20) implies

$$\text{either} \;\; \rho E_{\varphi} = 0 \;\; \text{or} \;\; \frac{\partial}{\partial \rho}\left(\gamma - \frac{\omega^2 \mu \varepsilon_{\varphi\varphi}}{\gamma}\right) = 0.$$

The case $\rho E_{\varphi} = 0$ implies that each component of the electromagnetic field vanishes.

The case $\frac{\partial}{\partial \rho}\left(\gamma - \frac{\omega^2 \mu \varepsilon_{\varphi\varphi}}{\gamma}\right) = 0$ implies that $\varepsilon_{\varphi\varphi}$ does not depend on ρ. This contradicts the initial assumption.

All this means that in a nonlinear waveguide (isotropic along axes ρ, φ) there are no nonlinear eigenmodes with polarization (9)! It should be noticed that in a linear waveguide there are eigenmodes for polarization (9).

Further we study only polarized waves (6) and (7) for nonlinear waveguides.

CHAPTER 10

TE WAVE PROPAGATION IN A LINEAR CIRCLE CYLINDRICAL WAVEGUIDE

§1. Statement of the Problem

Let us consider three-dimensional space $\mathbb{R}^3$ with Cartesian coordinate system $Oxyz$. The space is filled by isotropic medium with constant permittivity $\varepsilon \geq \varepsilon_0$, where ε_0 is the permittivity of free space. In this medium a circle cylindrical waveguide is placed. The waveguide is filled by anisotropic nonmagnetic medium. The waveguide has cross section $W := \left\{(x, y) : x^2 + y^2 < R^2\right\}$ and its generating line (the waveguide axis) is parallel to the axis Oz. We shall consider electromagnetic waves propagating along the waveguide axis, that is eigenmodes of the structure.

Let us consider also cylindrical coordinate system $O\rho\varphi z$. Axis Oz of cylindrical coordinate system coincides with axis Oz of Cartesian coordinate system.

The electromagnetic field depends on time harmonically [17]

$$\tilde{\mathbf{E}}\,(\rho,\ \varphi,\ z,\ t) = \mathbf{E}_+\,(\rho,\ \varphi,\ z)\cos\omega t + \mathbf{E}_-\,(\rho,\ \varphi,\ z)\sin\omega t,$$
$$\tilde{\mathbf{H}}\,(\rho,\ \varphi,\ z,\ t) = \mathbf{H}_+\,(\rho,\ \varphi,\ z)\cos\omega t + \mathbf{H}_-\,(\rho,\ \varphi,\ z)\sin\omega t,$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_+,\ \mathbf{E}_-,\ \tilde{\mathbf{H}},\ \mathbf{H}_+,\ \mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = \left(E_\rho, E_\varphi, E_z\right)^T, \quad \mathbf{H} = \left(H_\rho, H_\varphi, H_z\right)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential components on the media interfaces (on the boundary of the waveguide) and the radiation condition at infinity: the electromagnetic field exponentially decays as $\rho \to \infty$.

The permittivity inside the waveguide is the constant $\varepsilon = \varepsilon_2$.

The solutions to the Maxwell equations are sought in the entire space.

§2. TE Waves

Let us consider TE waves

$$\mathbf{E} = (0, E_\varphi, 0)^T, \quad \mathbf{H} = (H_\rho, 0, H_z)^T,$$

where $E_\varphi = E_\varphi(\rho, \varphi, z)$, $H_\rho = H_\rho(\rho, \varphi, z)$, and $H_z = H_z(\rho, \varphi, z)$.

Substituting the fields into Maxwell equations (1) we obtain

$$\begin{cases} \frac{1}{\rho}\frac{\partial H_z}{\partial \varphi} = 0, \\ \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -i\omega\varepsilon E_\varphi, \\ \frac{1}{\rho}\frac{\partial H_\rho}{\partial \varphi} = 0, \\ \frac{\partial E_\varphi}{\partial z} = -i\omega\mu H_\rho, \\ \frac{1}{\rho}\frac{\partial(\rho E_\varphi)}{\partial \rho} = i\omega\mu H_z. \end{cases}$$

It is obvious from the first and the third equations of this system that H_z and H_ρ do not depend on φ. This implies that E_φ does not depend on φ.

Waves propagating along waveguide axis Oz depend on z harmonically. This means that the fields components have the form

$$E_\varphi = E_\varphi(\rho)e^{i\gamma z}, H_\rho = H_\rho(\rho)e^{i\gamma z}, H_z = H_z(\rho)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system

$$\begin{cases} i\gamma H_\rho(\rho) - H_z'(\rho) = -i\omega\varepsilon E_\varphi(\rho), \\ i\gamma E_\varphi(\rho) = -i\omega\mu H_\rho(\rho), \\ \frac{1}{\rho}\left(\rho E_\varphi(\rho)\right)' = i\omega\mu H_z(\rho), \end{cases} \tag{2}$$

where $(\,\cdot\,)' \equiv \frac{d}{d\rho}$.

Then $H_z(\rho) = \frac{1}{i\omega\mu}\frac{1}{\rho}\left(\rho E_\varphi(\rho)\right)'$ and $H_\rho(\rho) = -\frac{\gamma}{\omega\mu}E_\varphi(\rho)$. From the first equation of system (2) we obtain

$$\left(\frac{1}{\rho}\left(\rho E_\varphi(\rho)\right)'\right)' + \left(\omega^2\mu\varepsilon - \gamma^2\right)E_\varphi(\rho) = 0.$$

Denoting by $u(\rho) := E_\varphi(\rho)$ we obtain

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + k^2 u = 0, \tag{3}$$

where

$$k^2 = \left(\omega^2\mu\varepsilon - \gamma^2\right),$$

and

$$\varepsilon = \begin{cases} \varepsilon_1, & \rho > R, \\ \varepsilon_2, & \rho < R. \end{cases}$$

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along waveguide axis, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of spectral parameter γ such that the nonzero real solution $u(\rho)$ to system (3) exists.

Note. We consider that γ is a real value, but in the linear case it is possible to consider the spectral parameter γ as a complex value. In nonlinear cases under our approach it is impossible to use complex value of γ.

Also we assume that function u is sufficiently smooth

$$u(x) \in C[0,+\infty) \cap C^1[0,R] \cap C^1\left[R,\ +\infty\right) \cap \\ \cap\, C^2(0,R) \cap C^2(R,+\infty).$$

Physical nature of the problem implies these conditions.

We will seek γ under conditions $\varepsilon_1 < \gamma^2 < \varepsilon_2$. It should be noticed that condition $\gamma^2 > \varepsilon_1$ holds if the value $\varepsilon_1 > 0$. If $\varepsilon_1 < 0$, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem

In the domain $\rho > R$ we have $\varepsilon = \varepsilon_1$. From (3) we obtain the equation $u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + k_1^2 u = 0$, where $k_1^2 = \omega^2\mu\varepsilon_1 - \gamma^2$. It is the Bessel equation and its general solution we take in the following form $u(\rho) = AH_1^{(1)}(k_1\rho) + +A_1H_1^{(2)}(k_1\rho)$, where $H_1^{(1)}$ and $H_1^{(2)}$ are the Hankel functions of first and second kind, respectively. Taking into account the condition at infinity we obtain the solution

$$u(\rho) = AH_1^{(1)}(k_1\rho), \tag{4}$$

where A is a constant. If $\operatorname{Re} k_1 = 0$, then[1]

$$u(\rho) = \tilde{A}K_1(|k_1|\rho), \tag{5}$$

where $K_1(z)$ is Macdonald function.

In the domain $\rho < R$ we have $\varepsilon = \varepsilon_2$. From (3) we obtain the equation $u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + k_2^2 u = 0$, where $k_2^2 = \omega^2\mu\varepsilon_2 - \gamma^2$. It is the Bessel equation and its general solution we take in the following form $u(\rho) = BJ_1(k_2\rho) + B_1N_1(k_2\rho)$, where J_1 and N_1 are Bessel and Neumann functions, respectively. The solution $u(\rho)$ is bounded at the point $\rho = 0$. This implies

$$u(\rho) = BJ_1(k_2\rho), \tag{6}$$

where B is a constant. If $\operatorname{Re} k_2 = 0$, then[1]

$$u(\rho) = \tilde{B}I_1(|k_2|\rho), \tag{7}$$

where $I_1(z)$ is the modified Bessel function.

[1]Since $H_1^{(1)}(iz) = -2\pi^{-1}K_1(z)$.

[1]Since $J_1(iz) = iI_1(z)$.

§4. Transmission Conditions and the Dispersion Equations

Tangential components of electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are E_φ and H_z. Hence we obtain

$$E_\varphi(R+0) = E_\varphi(R-0), \quad H_z(R+0) = H_z(R-0),$$

where the constant $E_\varphi^R := u(R) = E_\varphi(R+0)$ is supposed to be known (initial condition).

Further, we have $H_z(\rho) = \frac{1}{i\omega\mu}\left(\frac{1}{\rho}E_\varphi(\rho) + E'_\varphi(\rho)\right)$. Since $E_\varphi(\rho)$ and $H_z(\rho)$ are continuous at $\rho = R$; therefore, $E'_\varphi(\rho)$ is continuous at the point $\rho = R$. These conditions imply the transmission conditions for functions $u(\rho)$ and $u'(\rho)$

$$[u]_{\rho=R}, \ [u']_{\rho=R}, \tag{8}$$

where $[f]_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$ denotes a jump of the function f at the interface.

Taking into account solutions (5), (6) and transmission conditions (8), we obtain the dispersion equation

$$|k_1|K'_1(|k_1|R)J_1(k_2R) - k_2K_1(|k_1|R)J'_1(k_2R) = 0. \tag{9}$$

Using formulas (see [7])

$$J'_1(z) = J_0(z) - \frac{1}{z}J_1(z), \quad K'_1(z) = -K_0(z) - \frac{1}{z}K_1(z).$$

With the help of these formulas we obtain from (9) the dispersion equation in the final form

$$|k_1|K_0(|k_1|R)J_1(k_2R) + k_2K_1(|k_1|R)J_0(k_2R) = 0. \tag{10}$$

CHAPTER 11

TE WAVE PROPAGATION IN A CIRCLE CYLINDRICAL WAVEGUIDE WITH KERR NONLINEARITY

§1. Statement of the Problem

Let us consider three-dimensional space $\mathbb{R}^3$ with Cartesian coordinate system $Oxyz$. The space is filled by isotropic medium with constant permittivity $\varepsilon \geq \varepsilon_0$, where ε_0 is the permittivity of free space. In this medium a circle cylindrical waveguide is placed. The waveguide is filled by anisotropic nonmagnetic medium. The waveguide has cross section $W := \left\{(x, y) : x^2 + y^2 < R^2\right\}$ and its generating line (the waveguide axis) is parallel to the axis Oz. We shall consider electromagnetic waves propagating along the waveguide axis, that is eigenmodes of the structure.

Let us consider also cylindrical coordinate system $O\rho\varphi z$. Axis Oz of cylindrical coordinate system coincides with axis Oz of Cartesian coordinate system.

The electromagnetic field depends on time harmonically [17]

$$\begin{aligned}
\tilde{\mathbf{E}}(\rho,\ \varphi,\ z,\ t) &= \mathbf{E}_{+}(\rho,\ \varphi,\ z)\cos\omega t + \mathbf{E}_{-}(\rho,\ \varphi,\ z)\sin\omega t,\\
\tilde{\mathbf{H}}(\rho,\ \varphi,\ z,\ t) &= \mathbf{H}_{+}(\rho,\ \varphi,\ z)\cos\omega t + \mathbf{H}_{-}(\rho,\ \varphi,\ z)\sin\omega t,
\end{aligned}$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_{+},\ \mathbf{E}_{-},\ \tilde{\mathbf{H}},\ \mathbf{H}_{+},\ \mathbf{H}_{-}$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = (E_\rho, E_\varphi, E_z)^T, \quad \mathbf{H} = (H_\rho, H_\varphi, H_z)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot}\mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot}\mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential components on the media interfaces (on the boundary of the waveguide) and the radiation condition at infinity: the electromagnetic field exponentially decays as $\rho \to \infty$.

The permittivity inside the waveguide is described by Kerr law

$$\varepsilon = \varepsilon_2 + a|\mathbf{E}|^2,$$

where ε_2 is a constant part of the permittivity, a is the nonlinearity coefficient. The values ε_2 and a are supposed to be real constants.

It is necessary to find surface waves propagating along the waveguide axis.

The solutions to the Maxwell equations are sought in the entire space.

§2. TE Waves

Let us consider TE waves

$$\mathbf{E} = (0, E_\varphi, 0)^T, \quad \mathbf{H} = (H_\rho, 0, H_z)^T,$$

where $E_\varphi = E_\varphi(\rho, \varphi, z)$, $H_\rho = H_\rho(\rho, \varphi, z)$, and $H_z = H_z(\rho, \varphi, z)$.

Substituting the fields into Maxwell equations (1), we obtain

$$\begin{cases} \frac{1}{\rho}\frac{\partial H_z}{\partial \varphi} = 0, \\ \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -i\omega\varepsilon E_\varphi, \\ \frac{1}{\rho}\frac{\partial H_\rho}{\partial \varphi} = 0, \\ \frac{\partial E_\varphi}{\partial z} = -i\omega\mu H_\rho, \\ \frac{1}{\rho}\frac{\partial(\rho E_\varphi)}{\partial \rho} = i\omega\mu H_z. \end{cases}$$

It is obvious from the first and the third equations of this system that H_z and H_ρ do not depend on φ. This implies that E_φ does not depend on φ.

Waves propagating along waveguide axis Oz depend on z harmonically. This means that the fields components have the form

$$E_\varphi = E_\varphi(\rho)e^{i\gamma z}, H_\rho = H_\rho(\rho)e^{i\gamma z}, H_z = H_z(\rho)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system

$$\begin{cases} i\gamma H_\rho(\rho) - H_z'(\rho) = -i\omega\varepsilon E_\varphi(\rho), \\ i\gamma E_\varphi(\rho) = -i\omega\mu H_\rho(\rho), \\ \frac{1}{\rho}\left(\rho E_\varphi(\rho)\right)' = i\omega\mu H_z(\rho), \end{cases} \tag{2}$$

where $(\,\cdot\,)' \equiv \frac{d}{d\rho}$.

Then $H_z(\rho) = \frac{1}{i\omega\mu}\frac{1}{\rho}\left(\rho E_\varphi(\rho)\right)'$ and $H_\rho(\rho) = -\frac{\gamma}{\omega\mu}E_\varphi(\rho)$. From the first equation of system (2) we obtain

$$\left(\frac{1}{\rho}\left(\rho E_\varphi(\rho)\right)'\right)' + \left(\omega^2\mu\varepsilon - \gamma^2\right)E_\varphi(\rho) = 0.$$

Denoting by $u(\rho) := E_\varphi(\rho)$ we obtain

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + \left(\omega^2\mu\varepsilon - \gamma^2\right)u = 0, \tag{3}$$

and

$$\varepsilon = \begin{cases} \varepsilon_1, & \rho > R, \\ \varepsilon_2 + au^2, & \rho < R. \end{cases}$$

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along waveguide axis, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of spectral parameter γ such that the nonzero real solution $u(\rho)$ to system (3) exists.

Also we assume that function u is sufficiently smooth

$$u(x) \in C[0, +\infty) \cap C^1[0, +\infty) \cap C^2(0, R) \cap C^2(R, +\infty).$$

Physical nature of the problem implies these conditions.

We will seek γ under conditions $\varepsilon_1 < \gamma^2 < \varepsilon_2$. It should be noticed that condition $\gamma^2 > \varepsilon_1$ holds if the value $\varepsilon_1 > 0$. If $\varepsilon_1 < 0$, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem. Transmission Conditions

In the domain $\rho > R$ we have $\varepsilon = \varepsilon_1$. From (3) we obtain the equation

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + k_1^2 u = 0, \tag{4}$$

where $k_1^2 = \omega^2\mu\varepsilon_1 - \gamma^2$. It is the Bessel equation.

In the domain $\rho < R$ we have $\varepsilon = \varepsilon_2 + au^2$. From (3) we obtain the equation

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + k_2^2 u + \alpha u^3 = 0, \tag{5}$$

where $k_2^2 = \omega^2\mu\varepsilon_2 - \gamma^2$, $\alpha = a\omega^2\mu$. It is the nonlinear equation and its solution is unknown.

Tangential components of electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are E_φ and H_z. Hence we obtain

$$E_\varphi(R+0) = E_\varphi(R-0), \quad H_z(R+0) = H_z(R-0),$$

where the constant $E_\varphi^R := u(R) = E_\varphi(R+0)$ is supposed to be known (initial condition).

Further, we have $H_z(\rho) = \frac{1}{i\omega\mu}\left(\frac{1}{\rho}E_\varphi(\rho) + E'_\varphi(\rho)\right)$. Since $E_\varphi(\rho)$ and $H_z(\rho)$ are continuous at the point $\rho = R$; therefore, $E'_\varphi(\rho)$ is continuous at $\rho = R$. These conditions imply the transmission conditions for functions $u(\rho)$ and $u'(\rho)$

$$[u]_{\rho=R} = 0,\ [u']_{\rho=R} = 0, \tag{6}$$

where $[f]_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$ denotes a jump of the function f at the interface.

Let us formulate the transmission problem (*the problem* P). *It is necessary to find eigenvalues* γ *and corresponding to them nonzero eigenfunctions* $u(\rho)$ *such that* $u(\rho)$ *satisfy to equations* (4)–(5); *transmission conditions* (6) *and the radiation condition at infinity: eigenfunctions exponentially decay as* $\rho\to\infty$.

The general solution of equation (4) we take in the following form $u(\rho) = CH_1^{(1)}(k_1\rho) + C_1H_1^{(2)}(k_1\rho)$, where $H_1^{(1)}$ and $H_1^{(2)}$ are the Hankel functions of the first and the second kind, respectively.

In accordance with the radiation condition we obtain the solution

$$u(\rho) = C_1H_1^{(1)}(k_1\rho), \quad \rho > R, \tag{7}$$

where C_1 is a constant. If $\operatorname{Re} k_1 = 0$, then[1]

$$u(\rho) = \tilde{C}_1K_1(|k_1|\rho), \quad \rho > R, \tag{8}$$

where $K_1(z)$ is the Macdonald function. The radiation condition is fulfilled since $K_1(|k_1|\rho)\to 0$ as $\rho\to\infty$.

By $u(R+0) := E_0$ denote the field on the waveguide boundary. From formulas (7), (8) we obtain

$$u(\rho) = E_0\frac{H_1^{(1)}(k_1\rho)}{H_1^{(1)}(k_1R)} \tag{8*}$$

and

$$u(\rho) = E_0\frac{K_1(|k_1|\rho)}{K_1(|k_1|R)}. \tag{9*}$$

[1]Since $H_1^{(1)}(iz) = -\frac{2}{\pi}K_1(z)$.

§4. Nonlinear Integral Equation and Dispersion Equation

Consider nonlinear equation (5) written in the form

$$\rho u'' + u' + \left(k_2^2\rho - \frac{1}{\rho}\right)u + \alpha\rho u^3 = 0, \tag{9}$$

and the linear Bessel equation

$$\rho u'' + u' + \left(k_2^2\rho - \frac{1}{\rho}\right)u = 0. \tag{10}$$

This equation can be written in the operator form as

$$Lu = 0, \quad L = \rho\frac{d^2}{d\rho^2} + \frac{d}{d\rho} + \left(k_2^2\rho - \frac{1}{\rho}\right). \tag{11}$$

Let us consider the boundary problem for equation (10) with conditions (6). Construct the Green function G for the boundary value problem

$$LG = -\delta\left(\rho - s\right), \quad G\left|_{\rho=0}\right. = G'\left|_{\rho=R}\right. = 0 \quad (0 < s < R) \tag{12}$$

in the form[1] (see, e.g., [82])

$$G(\rho, s) = \\ = \begin{cases} \frac{\pi}{2}J_1(k_2\rho)\frac{J_1(k_2s)N_1'(k_2R) - J_1'(k_2R)N_1(k_2s)}{J_1'(k_2R)}, & \rho < s \leq R, \\ \frac{\pi}{2}J_1(k_2s)\frac{J_1(k_2\rho)N_1'(k_2R) - J_1'(k_2R)N_1(k_2\rho)}{J_1'(k_2R)}, & s < \rho \leq R. \end{cases} \tag{13}$$

The Green function exists if $J_1'(k_2R) \neq 0$.

Let us write equation (9) in the operator form

$$Lu + \alpha\rho u^3 = 0. \tag{14}$$

[1]Linearly independent solutions of the equation $Lu = 0$ satisfying the conditions $u|_{\rho=0} = u'|_{\rho=R} = 0$ are $u_1 = J_1(k_2\rho)$ and $u_2 = N_1'(k_2R)J_1(k_2\rho) - J_1'(k_2R)N_1(k_2\rho)$.

Using the second Green formula

$$\int_0^R (vLu - uLv)d\rho = \int_0^R \left(v\left(\rho u'\right)' - u\left(\rho v'\right)'\right)d\rho = \\ = R\left(u'(R)v(R) - v'(R)u(R)\right) \quad (15)$$

and assuming that $v = G$, we obtain

$$\int_0^R (GLu - uLG)d\rho = \\ = R\left(u'(R-0)G(R,s) - G'(R,s)u(R-0)\right) = \\ = Ru'(R-0)G(R,s), \quad (16)$$

since it is clear from (13) that $G'(R,s) = 0$.

From (12) we obtain

$$\int_0^R uLGd\rho = -\int_0^R u(\rho)\delta(\rho - s)d\rho = -u(s).$$

Further, using formula (14), we obtain

$$\int_0^R GLud\rho = -\alpha \int_0^R G(\rho,s)\rho u^3(\rho)d\rho.$$

Taking into account these results and formula (16), we obtain the nonlinear integral equation with respect to $u(s)$ ($u(\rho)$ is a solution of equation (5)) on interval $(0, R)$

$$u(s) = \alpha \int_0^R G(\rho,s)\rho u^3(\rho)d\rho + Ru'(R-0)G(R,s), \quad 0 \le s \le R. \quad (17)$$

It is easy to see that $G(R,s) = \frac{1}{k_2 R}\frac{J_1(k_2 s)}{J_1'(k_2 R)}$ (Wronskian of the Bessel and the Neumann functions proves this formula). Taking into account this result and using transmission conditions (6) we obtain from equation (17)

$$u(s) = \alpha \int_0^R G(\rho, s)\rho u^3(\rho) d\rho + u'(R+0)\frac{1}{k_2}\frac{J_1(k_2 s)}{J_1'(k_2 R)}, \quad 0 \le s \le R. \quad (18)$$

Using (9*) and denoting by $f(s) := E_0 \frac{|k_1|}{k_2}\frac{K_1'(|k_1|R)J_1(k_2 s)}{K_1(|k_1|R)J_1'(k_2 R)}$ we can rewrite equation (18) in the final form

$$u(s) = \alpha \int_0^R G(\rho, s)\rho u^3(\rho) d\rho + f(s), \quad 0 \le s \le R. \quad (19)$$

Using equation (19) and transmission conditions (6), we obtain the dispersion equation with respect to the propagation constants

$$u(R+0) = \alpha \int_0^R G(\rho, R)\rho u^3(\rho) d\rho + f(R).$$

Applying formula (9*) we obtain the DE in the following form

$$E_0 = \alpha \int_0^R G(\rho, R)\rho u^3(\rho) d\rho + f(R). \quad (20)$$

Let us denote by $N(\rho, s) := \alpha G(\rho, s)\rho$ and consider the integral equation (19)

$$u(s) = \int_0^R N(\rho, s) u^3(\rho) d\rho + f(s). \quad (21)$$

in $C[0, R]$ [67]. It is assumed that $f \in C[0, R]$ and $J_1'(k_2 R) \neq 0$. It is not difficult to see that the kernel $N(\rho, s)$ is continuous in the square $0 \le \rho,\ s \le R$.

Let us consider the linear integral operator

$$N\omega = \int_0^R N(\rho, s)\omega(\rho)d\rho \tag{22}$$

in $C[0, R]$. It is bounded, completely continuous, and satisfies the condition

$$\|N\| = \max_{s\in[0,R]} \int_0^R |N(\rho, s)|d\rho. \tag{23}$$

Since the nonlinear operator $B(u) = u^3(\rho)$ is bounded and continuous in $C[0, R]$; therefore, the nonlinear operator

$$F(u) = \int_0^R N(\rho, s)u^3(\rho)d\rho + f(s) \tag{24}$$

is completely continuous in any bounded set in $C[0, R]$.

In the subsequent reasoning, we need the following auxiliary number cubic equation

$$\|N\| \, r^3 + \|f\| = r, \tag{25}$$

where the norm $\|N\| > 0$ of the operator is defined by formula (23) and

$$\|f\| = \max_{s\in[0,R]} |f(s)|. \tag{26}$$

Consider the equation

$$r - \|N\| \, r^3 = \|f\| \tag{27}$$

and the function

$$y(r) := r - \|N\| \, r^3. \tag{28}$$

The function $y(r)$ has only one positive maximum point

$$r_{\max} = \frac{1}{\sqrt{3\,\|N\|}} \tag{29}$$

such that the value of the function at this point is

$$y_{\max} = y\left(r_{\max}\right) = \frac{2}{3\sqrt{3\left\|N\right\|}}. \tag{30}$$

Then, if $0 \le \|f\| \le \frac{2}{3\sqrt{3\|N\|}}$ equation (27) has two nonnegative roots r_* and r^*, $r_* \le r^*$, which satisfy the inequalities

$$0 \le r_* \le \frac{1}{\sqrt{3\left\|N\right\|}}, \quad \frac{1}{\sqrt{3\left\|N\right\|}} \le r^* \le \frac{1}{\sqrt{\left\|N\right\|}}. \tag{31}$$

These roots are easily found as solutions of the cubic equation

$$r^3 - \frac{1}{\sqrt{\|N\|}} r + \frac{\|f\|}{\|N\|} = 0. \tag{32}$$

According to [28], we have

$$r_* = -\frac{2}{\sqrt{3\left\|N\right\|}} \cos\left(\frac{\arccos\left(\frac{3\sqrt{3}}{2}\left\|f\right\|\sqrt{\left\|N\right\|}\right)}{3} - \frac{2\pi}{3}\right), \tag{33}$$

$$r^* = -\frac{2}{\sqrt{3\left\|N\right\|}} \cos\left(\frac{\arccos\left(\frac{3\sqrt{3}}{2}\left\|f\right\|\sqrt{\left\|N\right\|}\right)}{3} + \frac{2\pi}{3}\right). \tag{34}$$

If $\|f\| = 0$, then $r_* = 0$ and $r^* = \frac{1}{\sqrt{\|N\|}}$. If $0 < \|f\| < \frac{2}{3\sqrt{3\|N\|}}$, then $r_* < \frac{1}{\sqrt{3\|N\|}}$. For $\|f\| = \frac{2}{3}\frac{1}{\sqrt{3\|N\|}}$ we have $r_* = r^* = \frac{1}{\sqrt{3\|N\|}}$.

Thus we proved the following assertion.

Lemma 1. *If*

$$0 \le \|f\| < \frac{2}{3\sqrt{3\left\|N\right\|}}, \tag{35}$$

then equation (25) *has two nonnegative solutions* r_*, r^*; $r_* < r^*$.

Using the Schauder principle [67, 83], it can be shown that, for any $f \in S_{\tilde{r}}(0) \subset C[0, R]$, where $\tilde{r} = \frac{2}{3\sqrt{3\|N\|}}$, there exists a solution $u(\rho)$ to equation (19) in the ball $S^* = S_{r^*}(0)$.

Lemma 2. *If* $\|f\| \le \frac{2}{3\sqrt{3\|N\|}}$, *then equation* (19) *has at least one solution* $u(\rho)$ *such that* $\|u\| \le r^*$.

Proof. Since $F(u)$ is absolutely continuous, it is required only to verify whether F maps the ball S^* into itself. Suppose that $u \in S^*$. Using (22)–(24), we obtain

$$\|F\,(u)\| \le \|N\| \cdot \|u\|^3 + \|f\| \le \|N\|\,(r^*)^3 + \|f\| = r^*.$$

This implies that $FS^* \subset S^*$. The lemma is proved.

Now we prove that, if condition (27) holds, then equation (19) has a unique solution u in the ball $S_* = S_{r_*}$.

Theorem 1. *If* $\alpha \le A^2$, *where*

$$A = \frac{2}{3}\frac{1}{\|f\|\,\sqrt{3\,\|N_0\|}} \tag{36}$$

and

$$\|N_0\| = \max_{s\in[0,R]} \int_0^R |\rho G(\rho, s)|\, d\rho,$$

then equation (19) *has a unique continuous solution* $u \in C[0, R]$ *such that* $\|u\| \le r_*$.

Proof. If $u \in S_*$, then

$$\|F(u)\| \le \|N\| \cdot \|u\|^3 + \|f\| \le \|N\|\,(r_*)^3 + \|f\| = r_*.$$

If u_1, $u_2 \in S_*$, then

$$\|F(u_1) - F(u_2)\| = \left\| \int_0^R N(\rho, s)\left(u_1^3(\rho) - u_2^3(\rho)\right) d\rho \right\| \le$$
$$\le 3\,\|N\|\, r_*^2\, \|u_1 - u_2\|\,.$$

Since $\alpha \le A^2$, $f(s)$ satisfies condition (35). Hence inequality $r_* < \frac{1}{\sqrt{3\|N\|}}$ holds. Then, it follows that $3\,\|N\|\, r_*^2 < 1$.

Hence F maps S_* into itself and F is a contracting operator on S_*. Therefore equation (19) has a unique solution in S_*. The theorem is proved.

Note that $A > 0$ does not depend on α.

In what follows, we need the following assertion about the dependence of the solutions of the integral equation (19) on the parameter.

Theorem 2. *Let the kernel* N *and the right-hand side* f *of equation* (19) *depend continuously on the parameter* $\lambda \in \Lambda_0$, $N(\lambda, \rho, s) \subset C(\Lambda_0 \times [0, R] \times [0, R])$, $f(\lambda, s) \subset C(\Lambda_0 \times [0, R])$ *on some segment* Λ_0 *of the real number axis. Let also*

$$0 < \|f(\lambda)\| < \frac{2}{3\sqrt{3\,\|N(\lambda)\|}}. \tag{37}$$

Then, for $\lambda \in \Lambda_0$, *a unique solution* $u(\lambda, \rho)$ *of equation* (19) *exists and depends continuously on* λ, $u(\lambda, \rho) \subset C(\Lambda_0 \times [0, R])$.

Proof. Consider the equation

$$u(s, \lambda) = \int_0^R N(\lambda, \rho, s) u^3(\rho, \lambda) d\rho + f(s, \lambda). \tag{38}$$

Under the assumptions of the theorem, the existence and uniqueness of solutions $u(\lambda)$ follow from Theorem 1.

Let us prove that these solutions depend continuously on the parameter λ.

From (33) it immediately follows that $r_*(\lambda)$ depends continuously on λ in the segment Λ_0. Let $r_0 = \max\limits_{\lambda \in \Lambda_0} r_*(\lambda)$ and the maximum be achieved at the point λ_0, $r_*(\lambda) = r_0$.

Furthermore, let $Q = \max\limits_{\lambda \in \Lambda_0} \left(3r_*^2(\lambda)\,\|N(\lambda)\|\right)$ and the maximum be achieved at the point $\tilde{\lambda} \in \Lambda_0$, $Q = 3r_*^2(\tilde{\lambda}) \left\|N(\tilde{\lambda})\right\|$. Then, $Q < 1$ by virtue of assumption (37).

First, let us assume that $\|u(\lambda)\| \geq \|u(\lambda+\Delta\lambda)\|$. Then, the following inequalities are valid

$$|u(s,\lambda+\Delta\lambda)-u(s,\lambda)| =$$
$$= \left| \int\limits_0^R N(\lambda+\Delta\lambda,\rho,s)u^3(\rho,\lambda+\Delta\lambda)d\rho - \right.$$
$$\left. - \int\limits_0^R N(\lambda,\rho,s)u^3(\rho,\lambda)d\rho + (f(s,\lambda+\Delta\lambda)-f(s,\lambda)) \right| \leq$$
$$\leq \int\limits_0^R |N(\lambda+\Delta\lambda,\rho,s)-N(\lambda,\rho,s)| \cdot |u(\rho,\lambda+\Delta\lambda)|^3 \, d\rho +$$
$$+ \int\limits_0^R |N(\lambda,\rho,s)| \cdot \left|u^3(\rho,\lambda+\Delta\lambda)-u^3(\rho,\lambda)\right| d\rho +$$
$$+ |f(s,\lambda+\Delta\lambda)-f(s,\lambda)| \leq$$
$$\leq \|u(\lambda+\Delta\lambda)\|^3 \int\limits_0^R |N(\lambda+\Delta\lambda,\rho,s)-N(\lambda,\rho,s)| \, d\rho +$$
$$+ \|u(\lambda+\Delta\lambda)-u(\lambda)\| \times$$
$$\times \left(\|u(\lambda+\Delta\lambda)\|^2 + \|u(\lambda+\Delta\lambda)\| \cdot \|u(\lambda)\| + \|u(\lambda)\|^2 \right) \times$$
$$\times \int\limits_0^R |N(\lambda,\rho,s)| \, d\rho + \|f(\lambda+\Delta\lambda)-f(\lambda)\| \leq$$
$$\leq r_0^3 \|N(\lambda+\Delta\lambda)-N(\lambda)\| +$$
$$+ \|u(\lambda+\Delta\lambda)-u(\lambda)\| \, 3r_*^2(\lambda) \|N(\lambda)\| + \|f(\lambda+\Delta\lambda)-f(\lambda)\| .$$

Then, it follows that

$$\|u(\lambda+\Delta\lambda)-u(\lambda)\| \leq$$
$$\leq \frac{r_0^3 \|N(\lambda+\Delta\lambda)-N(\lambda)\| + \|f(\lambda+\Delta\lambda)-f(\lambda)\|}{1-3r_*^2(\lambda)\|N(\lambda)\|},$$

and

$$\|u(\lambda+\Delta\lambda)-u(\lambda)\| \leq$$
$$\leq \frac{r_0^3\,\|N(\lambda+\Delta\lambda)-N(\lambda)\|+\|f(\lambda+\Delta\lambda)-f(\lambda)\|}{1-Q}, \quad (39)$$

where Q and r_0 do not depend on λ.

Now, let $\|u(\lambda)\| < \|u(\lambda+\Delta\lambda)\|$. Then, all the preceding estimates remain valid if we replace λ by $\lambda+\Delta\lambda$, and $\lambda+\Delta\lambda$ by λ. Thus, estimate (39) also remains valid, which proves the theorem.

§5. Iteration Method

Approximate solutions u_n of integral equation (19) represented in the form $u = F(u)$ can be found by means of the iteration process $u_{n+1} = F(u_n)$, $n = 0,\ 1,\ ...,$

$$u_0 = 0,\ u_{n+1} = \alpha \int_0^R G(\rho, s)\rho u_n^3 d\rho + f,\ n = 0,\, 1,\, ... \quad (40)$$

The sequence u_n converges uniformly to the solution u of equation (19) by virtue of the fact that $F(u)$ is a contracting operator. The estimate of the convergence rate of iteration process (40) is also known. Let us formulate these results as the following proposition.

Proposition 1. *The sequence of approximate solutions* u_n *of equation* (19), *obtained by means of iteration process* (40) *converges in the norm of the space* $C[0, R]$ *to the* (*unique*) *exact solution* u *of this equation. The following estimate of the convergence rate is valid*

$$\|u_n - u\| \leq \frac{q^n}{1-q} f(u_0),\ n \to \infty, \quad (41)$$

where

$$q := 3Nr_*^2 < 1$$

is the coefficient of contraction of the mapping F.

§6. Theorem of Existence

Let us introduce the following dimensionless variables and parameters:[1] $\tilde{\rho} = k_0\rho$, $\tilde{z} = k_0 z$, $\tilde{R} = k_0 R$, $\tilde{\varepsilon} = \varepsilon/\varepsilon_0$, $\tilde{\mu} = \mu/\mu_0 = 1$, $\tilde{k}_2 = \sqrt{\tilde{\varepsilon}_2 - \tilde{\gamma}^2}$, $k_1 = \sqrt{\tilde{\gamma}^2 - \tilde{\varepsilon}_1}$ ($\tilde{\varepsilon}_2 > \tilde{\varepsilon}_1$), $\tilde{\gamma} = \gamma/k_0$, $\tilde{\alpha} = \alpha\tilde{C}_1^2/\varepsilon_0$, $\tilde{u} = u/\tilde{C}_1$, and $k_0 = \tilde{\omega}^2\varepsilon_0\tilde{\mu}_0$. Omitting the tilde and taking into account formulas

$$J_1'(k_2R) = J_0(k_2R) - (k_2R)^{-1}J_1(k_2R),$$
$$K_1'(|k_1|R) = -K_0(|k_1|R) - (k_1R)^{-1}K_1(|k_1|R),$$

we represent dispersion equation (20) in the normalized form

$$g(R,\gamma^2) = \alpha F(R,\gamma^2;u^3), \tag{42}$$

where

$$g(R,\gamma^2) = k_2RJ_0(k_2R)K_1(|k_1|R) + |k_1|RJ_1(k_2R)K_0(|k_1|R),$$
$$F(R,\gamma^2;u^3) = K_1(|k_1|R)\int\limits_0^R J_1(k_2\rho)\rho u^3(\rho,\gamma^2)d\rho.$$

The zeros of the function $\Phi(\gamma) \equiv g(\gamma) - \alpha F(\gamma)$ are those values of γ for which there exists a nontrivial solution of the problem P stated above. The following assertion gives us sufficient conditions for the existence of zeros of the function Φ.

Let j_{0m} be the m-th positive root of the Bessel function J_0; j_{1m} be the m-th positive root of the Bessel function J_1; j'_{1m} be the m-th positive root of the Bessel function J_1'; where $m = 1, 2, ...$

Then, we have

$j'_{11} = 1.841$, ...,	$j_{01} = 2.405$, ...,	$j_{11} = 3.832$, ...,
$j'_{12} = 5.331$, ...,	$j_{02} = 5.520$, ...,	$j_{12} = 7.016$, ...,
$j'_{13} = 8.536$, ...,	$j_{03} = 8.654$, ...,	$j_{13} = 10.173$, ...,
$j'_{14} = 11.706$, ...,	$j_{04} = 11.792$, ...,	$j_{14} = 13.324$, ...

[1] This problem depends on the initial condition E_0, see the note on p. 49.

Let us introduce the notation

$$\lambda_{1m} = \varepsilon_2 - j_{1m}^2/R^2; \quad \lambda_{2m} = \varepsilon_2 - j_{0m}^2/R^2, \quad m = 1, 2, \ \ldots;$$

and $\Lambda_i = [\lambda_{1i}, \lambda_{2i}]$, $\Lambda = \bigcup\limits_{i=1}^{m} \Lambda_i$.

Theorem 3. *Suppose that* $\varepsilon_2 > \varepsilon_1 > 0$, $0 < \alpha \leq \alpha_0$, *where*

$$\alpha_0 = \min \left(\min_{\lambda \in \Lambda} A(\lambda), \ \frac{\min\limits_{1 \leq l \leq 2, \ 1 \leq i \leq m} |g(\lambda_{li})|}{0.3 \cdot R^2 \left(\max\limits_{\lambda \in \Lambda} r_*(\lambda) \right)^3} \right), \tag{43}$$

and let the condition

$$\lambda_{1m} > \varepsilon_1 \tag{44}$$

hold for $m \geq 1$. *Then, there exist at least* m *values* γ_i, $i = \overline{1, m}$ *such that the problem* P *has a nonzero solution.*

Proof. Let $\lambda = \gamma^2$ and $\|u\| \leq r_* = r_*(\lambda)$.

Since $j'_{1i} \notin \Lambda$ for $i = 1, 2, 3, 4$, Green function (13) exists for $\gamma^2 \in \Lambda$. From (36) and the properties of the Green function, it follows that $A^2 = A^2(\lambda)$ is a continuous function on λ in the interval Λ, $\lambda \in \Lambda$. Let $A_0^2 = \min\limits_{\lambda \in \Lambda} A^2(\lambda)$ and $\alpha \leq A_0^2$. According to Theorem 1, there exists a unique continuous solution $u = u(\lambda)$ to equation (14) for any $\lambda \in \Lambda$ such that this solution is continuous and $\|u\| \leq r_* = r_*(\lambda)$. Let $r_0 = \max\limits_{\lambda \in \Lambda} r_*(\lambda)$. Since $|J_1(x)| \leq 0.6$ for nonnegative x, using the simplest estimate for integral $F(\lambda)$, we find that $|F(\lambda)| \leq 0.3 \cdot R^2 r_0^3$.

In view of the properties of the Macdonald functions, $K_0(x)$ and $K_1(x)$ are positive for positive x. The function $g(\lambda)$ is also continuous with respect to λ, and $g(\lambda_{1i})$, $g(\lambda_{2i}) < 0$, $i = 1$, ..., m. Thus, the equation $g(\lambda) = 0$ has a root λ_{0i} in the interval Λ_i, $\lambda_{1i} < \lambda_{0i} < \lambda_{2i}$.

Let us introduce the notation

$$M_1 := \min_{1 \leq i \leq m} |g(\lambda_{1i})|, \ \ M_2 := \min_{1 \leq i \leq m} |g(\lambda_{2i})|, \ \ M := \min\{M_1, M_2\};$$

$M > 0$ does not depend on α.

If $\alpha \leq \frac{M}{0.3 \cdot R^2 r_0^3}$, then

$$(g(\lambda_{1i}) - \alpha F(\lambda_{1i}))(g(\lambda_{2i}) - \alpha F(\lambda_{2i})) < 0.$$

Since the function $g(\lambda) - \alpha F(\lambda)$ is also continuous, the equation $g(\lambda) - \alpha F(\lambda) = 0$ has a root λ_i in the interval Λ_i, i.e. $\lambda_{1i} < \lambda_i < \lambda_{2i}$. We can select α_0 such that $\alpha_0 = \min\left\{A_0^2, \frac{M}{0.3 \cdot R^2 r_0^3}\right\}$. The theorem is proved.

From Theorem 3 it follows that, under the above assumptions, there exist axially symmetrical propagating TE waves in cylindrical dielectric waveguides of circular cross-section filled with a nonmagnetic isotropic medium with Kerr nonlinearity. This result generalizes the well-known similar statement for dielectric waveguides of circular cross-section filled with a linear medium (i.e., $\alpha = 0$).

§7. NUMERICAL METHOD

We present the numerical method for finding approximate solutions. In practise, as a rule, the propagation constants of the guiding system, i.e., the eigenvalues γ (or λ) for which there exist nontrivial solutions to the boundary value problem P, are of interest. Answers to the questions of the existence and localization of the eigenvalues γ are given in Theorem 3. Let us discuss the method for approximately finding the eigenvalues γ.

Let eigenvalues γ be sought in the interval $[A_1, A_2]$ (this interval can be selected based on the results of Theorem 3 or from practical considerations). Let us introduce a grid in this interval with the nodes $\gamma^{(j)} = A_1 + j(A_2 - A_1)/N$, $j = 0, \ldots, N$, where N satisfies the condition $A_2 - A_1 < N\delta$ and δ is the desired accuracy of the eigenvalues γ. The values of the function Φ are computed at the nodes $\gamma^{(j)}$ by solving the integral equation (19) for each $\gamma^{(j)}$ by means of iteration algorithm (40) with the desired accuracy. Next, it is determined whether there is the signs' reversal in the sequence of numbers $\Phi\left(\gamma^{(j)}\right)$. If $\Phi\left(\gamma^{(j)}\right)\Phi\left(\gamma^{(j+1)}\right) < 0$, then we approximately set $\gamma = \left(\gamma^{(j)} + \gamma^{(j+1)}\right)/2$.

CHAPTER 12

TM WAVE PROPAGATION IN A LINEAR CIRCLE CYLINDRICAL WAVEGUIDE

§1. Statement of the Problem

Let us consider three-dimensional space $\mathbb{R}^3$ with Cartesian coordinate system $Oxyz$. The space is filled by isotropic medium with constant permittivity $\varepsilon \geq \varepsilon_0$, where ε_0 is the permittivity of free space. In this medium a circle cylindrical waveguide is placed. The waveguide is filled by anisotropic nonmagnetic medium. The waveguide has cross section $W := \left\{(x, y) : x^2 + y^2 < R^2\right\}$ and its generating line (the waveguide axis) is parallel to the axis Oz. We shall consider electromagnetic waves propagating along the waveguide axis, that is eigenmodes of the structure.

Let us consider also cylindrical coordinate system $O\rho\varphi z$. Axis Oz of cylindrical coordinate system coincides with axis Oz of Cartesian coordinate system.

Electromagnetic field depends on time harmonically [17]

$$\tilde{\mathbf{E}}\,(\rho,\ \varphi,\ z,\ t) = \mathbf{E}_+\,(\rho,\ \varphi,\ z)\cos\omega t + \mathbf{E}_-\,(\rho,\ \varphi,\ z)\sin\omega t,$$
$$\tilde{\mathbf{H}}\,(\rho,\ \varphi,\ z,\ t) = \mathbf{H}_+\,(\rho,\ \varphi,\ z)\cos\omega t + \mathbf{H}_-\,(\rho,\ \varphi,\ z)\sin\omega t,$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_+,\ \mathbf{E}_-,\ \tilde{\mathbf{H}},\ \mathbf{H}_+,\ \mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of the electromagnetic field

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = \left(E_\rho, E_\varphi, E_z\right)^T, \quad \mathbf{H} = \left(H_\rho, H_\varphi, H_z\right)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields is a function of three spatial variables.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot}\mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot}\mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential components on the media interfaces (on the boundary of the waveguide) and the radiation condition at infinity: the electromagnetic field exponentially decays as $\rho \to \infty$.

The permittivity inside the waveguide is described by the diagonal tensor

$$\tilde{\varepsilon} = \begin{pmatrix} \varepsilon_{\rho\rho} & 0 & 0 \\ 0 & \varepsilon_{\varphi\varphi} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix},$$

where $\varepsilon_{\rho\rho}$, ε_{zz} are constants. For TM wave it does not matter what form $\varepsilon_{\varphi\varphi}$ has, as it is not contained in the equations below.

The solutions to the Maxwell equations are sought in the entire space.

§2. TM Waves

Let us consider TM waves

$$\mathbf{E} = (E_\rho, 0, E_z)^T, \quad \mathbf{H} = (0, H_\varphi, 0)^T,$$

where $E_\rho = E_\rho(\rho, \varphi, z)$, $E_z = E_z(\rho, \varphi, z)$, and $H_\varphi = H_\varphi(\rho, \varphi, z)$.

Substituting the fields into Maxwell equations (1) we obtain

$$\begin{cases} \dfrac{\partial H_\varphi}{\partial z} = i\omega\varepsilon_{\rho\rho}E_\rho, \\ \dfrac{1}{\rho}\dfrac{\partial(\rho H_\varphi)}{\partial\rho} = -i\omega\varepsilon_{zz}E_z, \\ \dfrac{1}{\rho}\dfrac{\partial E_z}{\partial\varphi} = 0, \\ \dfrac{\partial E_\rho}{\partial z} - \dfrac{\partial E_z}{\partial\rho} = i\omega\mu H_\varphi, \\ \dfrac{1}{\rho}\dfrac{\partial E_\rho}{\partial\varphi} = 0. \end{cases}$$

It is obvious from the third and the fifth equations of this system that E_z and E_ρ do not depend on φ. This implies that H_φ does not depend on φ.

Waves propagating along waveguide axis Oz depend on z harmonically. This means that the fields components have the form

$$E_\rho = E_\rho(\rho)e^{i\gamma z}, E_z = E_z(\rho)e^{i\gamma z}, H_\varphi = H_\varphi(\rho)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system

$$\begin{cases} \gamma H_\varphi = \omega\varepsilon_{\rho\rho}E_\rho, \\ \frac{1}{\rho}(\rho H_\varphi)' = -i\omega\varepsilon_{zz}E_z, \\ i\gamma E_\rho - E_z' = i\omega\mu H_\varphi, \end{cases} \tag{2}$$

where $(\,\cdot\,)' \equiv \frac{d}{d\rho}$.

Using the third equation of system (2) it is easy to find that $H_\varphi(\rho) = \frac{1}{i\omega\mu}\left(i\gamma E_\rho(\rho) - E_z'(\rho)\right)$. And system (2) implies

$$\begin{cases} \frac{\gamma}{\omega\mu}\left(\gamma E_\rho(\rho) + (iE_z)'(\rho)\right) = \omega\varepsilon_{\rho\rho}E_\rho(\rho), \\ \frac{1}{\omega\mu}\frac{1}{\rho}\left(\gamma\rho E_\rho(\rho) + \rho(iE_z)'(\rho)\right)' = -\omega\varepsilon_{zz}(iE_z). \end{cases}$$

It is convenient to suppose that $\varepsilon_{\rho\rho} = \varepsilon_0\varepsilon_\rho$ and $\varepsilon_{zz} = \varepsilon_0\varepsilon_z$, where ε_0 is the permittivity of free space.

Denoting by $u_1(\rho) := E_\rho(\rho)$, $u_2(\rho) := iE_z(\rho)$ and $k_0^2 := \omega^2\mu\varepsilon_0$, from the latter system we obtain (we omit the independent variable if it does not confuse)

$$\begin{cases} \gamma^2 u_1 + \gamma u_2' = k_0^2\varepsilon_\rho u_1, \\ \frac{\gamma}{\rho}\left(\rho u_1\right)' + \frac{1}{\rho}\left(\rho u_2'\right)' = -k_0^2\varepsilon_z u_2. \end{cases}$$

From the first equation we find $u_1 = \frac{\gamma}{k_0^2\varepsilon_\rho - \gamma^2}u_2'$. Then, from the second equation we obtain $\frac{1}{\rho}\left(\rho u_2'\right)' + \frac{\varepsilon_z}{\varepsilon_\rho}\left(k_0^2\varepsilon_\rho - \gamma^2\right)u_2 = 0$.

Thus inside the waveguide the functions u_1 and u_2 are defined from the system

$$\begin{cases} u_1 = \frac{\gamma}{k_\rho^2}u_2', \\ \frac{1}{\rho}\left(\rho u_2'\right)' + \frac{\varepsilon_z}{\varepsilon_\rho}k_\rho^2 u_2 = 0, \end{cases} \tag{3}$$

where $k_\rho^2 = k_0^2\varepsilon_\rho - \gamma^2$.

It is clear that outside the waveguide the functions u_1 and u_2 are defined from system (3), where $\varepsilon_{zz} = \varepsilon_{\rho\rho} = \varepsilon_1$

$$\begin{cases} u_1 = -\frac{\gamma}{k_1^2} u_2', \\ \frac{1}{\rho}\left(\rho u_2'\right)' - k_1^2 u_2 = 0, \end{cases} \tag{4}$$

where $k_1^2 = \gamma^2 - k_0^2\varepsilon_1$.

The second equations of system (3) and (4) the Bessel equations.

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along waveguide axis, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of spectral parameter γ, such that the nonzero real solutions $u_1(\rho)$ and $u_2(\rho)$ to systems (3) and (4) exist.

Note. We consider that γ is a real value, but in the linear case it is possible to consider this spectral parameter γ as a complex value. In nonlinear cases under our approach it is impossible to use complex value of γ.

Also we assume that functions u_1 and u_2 are sufficiently smooth

$$\begin{aligned} &u_1(x) \in C[0,R] \cap C\,[R\,,+\infty) \cap C^1[0,R] \cap C^1\,[R,\ +\infty)\,, \\ &u_2(x) \in C\,[0\,,+\infty) \cap C^1[0,R] \cap C^1\,[R,\ +\infty) \cap \\ &\qquad \cap\, C^2(0,R) \cap C^2(R,+\infty). \end{aligned}$$

Physical nature of the problem implies these conditions.

We seek γ under conditions $\varepsilon_1 < \gamma^2 < \varepsilon_2$. It should be noticed that condition $\gamma^2 > \varepsilon_1$ holds if the value $\varepsilon_1 > 0$. If $\varepsilon_1 < 0$, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem

In the domain $\rho < R$ we write the solutions of system (3) in the following form [39]

$$\begin{cases} u_1(\rho) = \frac{\gamma}{k_\rho^2}\sqrt{\beta}\left(C_1 J_0'\left(\sqrt{\beta}\rho\right) + C_2 N_0'\left(\sqrt{\beta}\rho\right)\right), \\ u_2(\rho) = C_1 J_0\left(\sqrt{\beta}\rho\right) + C_2 N_0\left(\sqrt{\beta}\rho\right), \end{cases}$$

where $\beta = \frac{\varepsilon_z}{\varepsilon_\rho} k_\rho^2$.

The functions J_0 and N_0 are the Bessel and the Neumann functions of zero-orders, respectively. The Neumann function $N_0(\rho)$ has a singularity at the point $\rho = 0$. On the other hand, it is clear that the field intensity is bounded inside the waveguide. Taking this and formula $J_0'(z) = -J_1(z)$ [21] into account we obtain

$$\begin{cases} u_1(\rho) = -\frac{\gamma}{k_\rho^2}\sqrt{\beta}C_1 J_1\left(\sqrt{\beta}\rho\right), \\ u_2(\rho) = C_1 J_0\left(\sqrt{\beta}\rho\right), \end{cases} \tag{5}$$

where $\beta = \frac{\varepsilon_z}{\varepsilon_\rho}k_\rho^2$.

In the domain $\rho > R$ we write the solutions of system (4) in the following form [39]

$$\begin{cases} u_1(\rho) = -\frac{\gamma}{k_1}\left(C_3 I_0'(k_1\rho) + C_4 K_0'(k_1\rho)\right), \\ u_2(\rho) = C_3 I_0(k_1\rho) + C_4 K_0(k_1\rho). \end{cases}$$

The functions I_0 and K_0 are the modified Bessel functions (the Bessel functions of imaginary argument). The function $I_0(\rho)$ tends to infinity as $\rho \to +\infty$, and the function $K_0(\rho)$ tends to zero as $\rho \to +\infty$. Taking the formula $K_0'(z) = -K_1(z)$ [21] into account we obtain

$$\begin{cases} u_1(\rho) = \frac{\gamma}{k_1}C_4 K_1(k_1\rho), \\ u_2(\rho) = C_4 K_0(k_1\rho). \end{cases} \tag{6}$$

§4. Transmission Conditions and Dispersion Equation

Tangential components of electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are E_z and E_φ Hence we obtain

$$E_z(R+0) = E_z(R-0), \quad H_\varphi(R+0) = H_\varphi(R-0),$$

where the constant $E_z^R := u_2(R) = E_z(R+0)$ is supposed to be known (initial condition).

It is well known that a normal component of an electromagnetic field has a jump at the medium interface. In this case the normal component is E_ρ. It is also well known that the value εE_ρ is continuous at the medium interface. From all above we obtain the transmission conditions for the functions u_1 and u_2

$$[\tilde{\varepsilon} u_1]_{\rho=R} = 0, \ [u_2]_{\rho=R} = 0, \tag{7}$$

where $[f]_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$ denotes a finite jump of the function f at the interface, $\tilde{\varepsilon} = \varepsilon_1$ when $\rho > R$, and $\tilde{\varepsilon} = \varepsilon_z$ when $\rho < R$.

Taking into account solutions (5), (6) and transmission conditions (7), we obtain the dispersion equation

$$\varepsilon_z k_1 J_1\left(\sqrt{\beta}R\right) K_0(k_1 R) + \varepsilon_1 \sqrt{\beta} J_0\left(\sqrt{\beta}R\right) K_1(k_1 R) = 0, \tag{8}$$

where $\beta = \frac{\varepsilon_z}{\varepsilon_\rho} k_\rho^2$, $k_\rho^2 = k_0^2 \varepsilon_\rho - \gamma^2$, $k_1^2 = \gamma^2 - k_0^2 \varepsilon_1$, and $k_0^2 = \omega^2 \mu \varepsilon_0$.

It should be noticed that equation (8) can be used to study metamaterials.

In the case of isotropic waveguide, i.e. when $\varepsilon_2 := \varepsilon_\rho = \varepsilon_z$ and $k_2^2 := k_\rho^2 = k_0^2 \varepsilon_2 - \gamma^2$ we obtain the well-known dispersion equation (see [64])

$$\varepsilon_2 k_1 J_1 (k_2 R) K_0(k_1 R) + \varepsilon_1 k_2 J_0 (k_2 R) K_1(k_1 R) = 0. \tag{9}$$

CHAPTER 13

TM WAVE PROPAGATION IN A CIRCLE CYLINDRICAL WAVEGUIDE WITH KERR NONLINEARITY

§1. Statement of the Problem

Let us consider three-dimensional space $\mathbb{R}^3$ with Cartesian coordinate system $Oxyz$. The space is filled by isotropic medium with constant permittivity $\varepsilon \geq \varepsilon_0$, where ε_0 is the permittivity of free space. In this medium a circle cylindrical waveguide is placed. The waveguide is filled by anisotropic nonmagnetic medium. The waveguide has cross section $W := \left\{(x, y) : x^2 + y^2 < R^2\right\}$ and its generating line (the waveguide axis) is parallel to the axis Oz. We shall consider electromagnetic waves propagating along the waveguide axis, that is eigenmodes of the structure.

Let us consider also cylindrical coordinate system $O\rho\varphi z$ and axis Oz of cylindrical coordinate system coincides with axis Oz of Cartesian coordinate system.

Electromagnetic field depends on time harmonically [17]:

$$\tilde{\mathbf{E}}\,(\rho,\ \varphi,\ z,\ t) = \mathbf{E}_+\,(\rho,\ \varphi,\ z)\cos\omega t + \mathbf{E}_-\,(\rho,\ \varphi,\ z)\sin\omega t,$$
$$\tilde{\mathbf{H}}\,(\rho,\ \varphi,\ z,\ t) = \mathbf{H}_+\,(\rho,\ \varphi,\ z)\cos\omega t + \mathbf{H}_-\,(\rho,\ \varphi,\ z)\sin\omega t,$$

where ω is the circular frequency; $\tilde{\mathbf{E}},\ \mathbf{E}_+,\ \mathbf{E}_-,\ \tilde{\mathbf{H}},\ \mathbf{H}_+,\ \mathbf{H}_-$ are real functions. Everywhere below the time multipliers are omitted.

Let us form complex amplitudes of electromagnetic field $(\mathbf{E}, \mathbf{H})$:

$$\mathbf{E} = \mathbf{E}_+ + i\mathbf{E}_-,$$
$$\mathbf{H} = \mathbf{H}_+ + i\mathbf{H}_-,$$

where

$$\mathbf{E} = \left(E_\rho, E_\varphi, E_z\right)^T, \quad \mathbf{H} = \left(H_\rho, H_\varphi, H_z\right)^T,$$

and $(\,\cdot\,)^T$ denotes the operation of transposition and each component of the fields $\mathbf{E}$, $\mathbf{H}$ is a function of three spatial variables.

Electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

the continuity condition for the tangential components on the media interfaces (on the boundary of the waveguide) and the radiation condition at infinity: the electromagnetic field exponentially decays as $\rho \to \infty$.

The permittivity inside the waveguide is described by Kerr law

$$\varepsilon = \varepsilon_0 \left(\varepsilon_2 + \alpha|\mathbf{E}|^2\right),$$

where ε_2 is a constant part of the permittivity, α is the nonlinearity coefficient. The values ε_2 and α are supposed to be real constants.

It is necessary to find surface waves propagating along the waveguide axis.

The solutions to the Maxwell equations are sought in the entire space.

§2. TM Waves

Let us consider TM waves

$$\mathbf{E} = (E_\rho, 0, E_z)^T, \quad \mathbf{H} = (0, H_\varphi, 0)^T,$$

where $E_\rho = E_(\rho, \varphi, z)$, $E_z = E_z(\rho, \varphi, z)$, and $H_\varphi = H_\varphi(\rho, \varphi, z)$.

Substituting the fields $\mathbf{E}$ and $\mathbf{H}$ into Maxwell equations (1), we obtain

$$\begin{cases} \frac{1}{\rho}\frac{\partial E_z}{\partial \varphi} = 0, \\ \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = i\omega\mu H_\varphi, \\ \frac{1}{\rho}\frac{\partial E_\rho}{\partial \varphi} = 0, \\ \frac{\partial H_\varphi}{\partial z} = i\omega\varepsilon E_\rho, \\ \frac{1}{\rho}\frac{\partial(\rho H_\varphi)}{\partial \rho} = -i\omega\varepsilon E_z. \end{cases}$$

It is obvious from the first and the third equations of this system that E_z and E_ρ do not depend on φ. This implies that H_φ does not depend on φ.

Waves propagating along waveguide axis Oz depend on z harmonically. This means that the components of the field $(\mathbf{E}, \mathbf{H})$ have the form

$$E_\rho = E_\rho(\rho;\gamma)e^{i\gamma z}, E_z = E_z(\rho;\gamma)e^{i\gamma z}, H_\varphi = H_\varphi(\rho;\gamma)e^{i\gamma z},$$

where γ is the propagation constant (the spectral parameter of the problem).

So we obtain from the latter system

$$\begin{cases} i\gamma E_\rho(\rho) - E_z'(\rho) = i\omega\mu H_\varphi(\rho), \\ i\gamma H_\varphi(\rho) = i\omega\varepsilon E_\rho(\rho), \\ \frac{1}{\rho}\left(\rho H_\varphi(\rho)\right)' = -i\omega\varepsilon E_z(\rho), \end{cases} \tag{2}$$

where $(\,\cdot\,)' \equiv \frac{d}{d\rho}$.

From the first equation of system (2) we obtain

$$H_\varphi(\rho) = \frac{1}{i\omega\mu}\left(i\gamma E_\rho(\rho) - E_z'\right). \tag{3}$$

Substituting expression (3) into equations of system (2), we obtain

$$\begin{cases} -\gamma\left(iE_z\right)' = \left(\gamma^2 - \omega^2\varepsilon\mu\right)E_\rho, \\ -\gamma\frac{1}{\rho}\left(\rho E_\rho\right)' - \frac{1}{\rho}\left(\rho\left(iE_z\right)'\right)' = \omega^2\varepsilon\mu\left(iE_z\right). \end{cases} \tag{4}$$

Introduce the notation

$$E_\rho(\rho;\gamma) = u_1(\rho,\gamma),\ iE_z(\rho;\gamma) = u_2(\rho,\gamma). \tag{5}$$

Outside and inside the waveguide $\varepsilon = \tilde{\varepsilon}\varepsilon_0$, where

$$\tilde{\varepsilon} = \begin{cases} \varepsilon_1, & \rho > R; \\ \varepsilon_2 + \alpha\left(u_1^2 + u_2^2\right), & \rho < R, \end{cases}$$

and also let $k_0^2 = \omega^2\varepsilon_0\mu$, where $k_0 > 0$ is the wave number of free space.

We consider that $u_1(\rho;\gamma)$, $u_2(\rho;\gamma)$ are real functions. We will omit argument(s) γ and/or ρ if there are no misunderstandings.

Using (5), from system (4) we obtain

$$\begin{cases} \gamma u_2' + \left(\gamma^2 - k_0^2\tilde{\varepsilon}\right) u_1 = 0, \\ \gamma \frac{1}{\rho}(\rho u_1)' + \frac{1}{\rho}\left(\rho u_2'\right)' + k_0^2\tilde{\varepsilon} u_2 = 0. \end{cases} \tag{6}$$

It is necessary to find eigenvalues γ of the problem that correspond to surface waves propagating along waveguide axis, i.e., the eigenvalues corresponding to the eigenmodes of the structure. We seek the real values of spectral parameter γ, such that the nonzero real solutions u_1 and u_2 to system (6) exist. We suppose that γ is a real value (due to $|\mathbf{E}|^2$ does not depend on z, see the footnote on p. 33, and also the note on p. 198).

Also we assume that functions u_1 and u_2 are sufficiently smooth

$$\begin{aligned} &u_1(x) \in C[0,R] \cap C\,[R\,,+\infty) \cap C^1[0,R] \cap C^1\,[R,\ +\infty)\,, \\ &u_2(x) \in C\,[0\,,+\infty) \cap C^1[0,R] \cap C^1\,[R,\ +\infty) \cap \\ &\qquad \cap\, C^2(0,R) \cap C^2(R,+\infty). \end{aligned}$$

Physical nature of the problem implies these conditions.

We will seek γ under conditions $\varepsilon_1 < \gamma^2 < \varepsilon_2$. It should be noticed that condition $\gamma^2 > \varepsilon_1$ holds if $\varepsilon_1 > 0$. If $\varepsilon_1 < 0$, then $\gamma^2 > 0$.

§3. Differential Equations of the Problem. Transmission Conditions

In the domain $\rho > R$ we have $\varepsilon = \varepsilon_1\varepsilon_0$. System (6) takes the form

$$\begin{cases} k_1^2 u_1 + \gamma u_2' = 0, \\ -\gamma \frac{1}{\rho}(\rho u_1)' - \frac{1}{\rho}\left(\rho u_2'\right)' - k_0^2\varepsilon_1 u_2 = 0, \end{cases} \tag{7}$$

where $k_1^2 = \gamma^2 - k_0^2\varepsilon_1$.

From the first equation of this system we have $u_1 = -\frac{\gamma}{k_1^2}u_2'$. Substituting this expression into the second equation of this system we obtain the following equation with respect to u_2

$$\frac{\gamma^2}{k_1^2}\frac{1}{\rho}\left(\rho u_2'\right)' - \frac{1}{\rho}\left(\rho u_2'\right)' - k_0^2\varepsilon_1 u_2 = 0.$$

After simple transformation this equation is reduced to the Bessel equation

$$\frac{1}{\rho}\left(\rho u_2'\right)' - k_1^2 u_2 = 0. \tag{8}$$

Denoting by $k_2^2 := k_0^2\varepsilon_2 - \gamma^2$, from (6) we obtain the system of equations inside the waveguide

$$\begin{cases} -k_2^2 u_1 + \gamma u_2' = f_1, \\ -\gamma \frac{1}{\rho}(\rho u_1)' - \frac{1}{\rho}\left(\rho u_2'\right)' - k_0^2\varepsilon_2 u_2 = f_2, \end{cases} \tag{9}$$

where

$$f_1 = k_0^2\alpha \left|\mathbf{u}\right|^2 u_1, \quad f_2 = k_0^2\alpha \left|\mathbf{u}\right|^2 u_2$$

and

$$\left|\mathbf{u}\right|^2 = u_1^2 + u_2^2, \quad \mathbf{u} = (u_1, u_2)^T.$$

Tangential components of electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are E_z and H_φ. Hence, we obtain

$$E_z(R+0) = E_z(R-0),\ H_\varphi(R+0) = H_\varphi(R-0),$$

where the constant $E_z^R = u_2(R) = E_z(R+0)$ is supposed to be known (initial condition).

It is well known that a normal component of electromagnetic field have a finite jump at the medium interface. In this case the normal component is E_ρ. It is also well known that the value $\tilde{\varepsilon}E_\rho$ is continuous at the medium interface. Thus we obtain the transmission conditions for the functions u_1 and u_2:

$$\left[\tilde{\varepsilon}u_1\right]_{\rho=R} = 0,\ \left[u_2\right]_{\rho=R} = 0, \tag{10}$$

where $[f]_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$ denotes a finite jump of the function f at the interface.

From the first condition (10) we obtain

$$\varepsilon_2 u_1|_{\rho=R-0} - \varepsilon_1 u_1|_{\rho=R+0} + \alpha u_1 |\mathbf{u}|^2\Big|_{\rho=R-0} = 0. \tag{11}$$

Let us formulate the transmission problem (*the problem P*). *It is necessary to find eigenvalues γ and corresponding nonzero eigenfunctions $u_1(\rho)$ and $u_2(\rho)$ such that $u_1(\rho)$ and $u_2(\rho)$ satisfy to the continuity conditions (see §2), satisfy to system* (9) *on* $(0, R)$, *satisfy to system* (7) *on* $(R, +\infty)$; *transmission conditions* (10) *and the radiation condition at infinity: eigenfunctions exponentially decay as $\rho \to \infty$. The spectral parameter of the problem is the real value γ.*

In accordance with the condition at infinity the solution of system (7) has the form

$$u_1 \equiv E_\rho = -\frac{\gamma}{k_1} C K_0'(k_1\rho), \; u_2 \equiv E_z = C K_0(k_1\rho), \qquad (12)$$

where C is an arbitrary constant; $K_0(z) = \frac{\pi i}{2} H_0^{(1)}(iz)$ is the Macdonald function [21].

It should be noticed that formulating *the problem P* it is enough to require only boundedness of the function $u_1(\rho)$, and $u_2(\rho)$ at infinity instead of exponential decaying. Indeed, the geeral solution of equation (8) is a linear combination of two cylinder functions [82], one of them (the Macdonald function $K_0(z)$) exponentially decays at infinity, and the other one exponentially increases at infinity. Hereupon the other one must be neglected by virtue of the solutions boundedness. Thus any bounded solution $u_1(\rho)$ and $u_2(\rho)$ of system (7) exponentially decays at infinity.

§4. Nonlinear Integral Equation and Dispersion Equation

Let us consider nonlinear system (9). From the first equation of this system we obtain[1]

$$u_1 = \frac{1}{k_2^2}(\gamma u_2' - f_1). \qquad (13)$$

Substituting (13) it to the second equation of the system (9), we have $-\gamma \frac{1}{\rho}\left(\rho \frac{1}{k_2^2}(\gamma u_2' - f_1)\right)' - \frac{1}{\rho}(\rho u_2')' - k_0^2 \varepsilon_2 u_2 = f_2$. It can be

[1] We recall that f_1 depends on u_1.

written as

$$Lu_2 \equiv \left(\rho u_2'\right)' + k_2^2 \rho u_2 = \frac{k_2^2}{k_0^2 \varepsilon_2} \left(\frac{\gamma}{k_2^2} (\rho f_1)' - \rho f_2 \right) \tag{14}$$

with linear part $Lu_2 \equiv (\rho u_2')' + k_2^2 \rho u_2$.

With the help of the corresponding Green function one can invert the linear part (the differential operator L) and obtain more convenient to studying an integro-differential equation.

Equation (14) can be also written in the form

$$\left(\rho u_2'\right)' + k_2^2 \rho u_2 = W,\ 0 < \rho < R, \tag{15}$$

where

$$W(\rho) = \frac{k_2^2}{k_0^2 \varepsilon_2} \left(\frac{\gamma}{k_2^2} (\rho f_1)' - \rho f_2 \right).$$

Let us construct the Green function for the boundary value problem

$$\begin{cases} LG = \delta(\rho - s), \\ G|_{\rho=0} \text{is bounded}, G|_{\rho=R} = 0,\ 0 < s < R, \end{cases} \tag{16}$$

and the differential operator is defined by formula

$$L = \rho \frac{d^2}{d\rho^2} + \frac{d}{d\rho} + k_2^2 \rho.$$

The Green function can be obtained in the following form [82]

$$G(\rho, s) = \\ = \begin{cases} \frac{\pi}{2} J_0(k_2 \rho) \frac{N_0(k_2 s) J_0(k_2 R) - J_0(k_2 s) N_0(k_2 R)}{J_0(k_2 R)}, & \rho < s \le R, \\ \frac{\pi}{2} J_0(k_2 s) \frac{N_0(k_2 \rho) J_0(k_2 R) - J_0(k_2 \rho) N_0(k_2 R)}{J_0(k_2 R)}, & s < \rho \le R, \end{cases} \tag{17}$$

where $J_0(\rho)$ is the zero-order Bessel function; $N_0(\rho)$ is the zero-order Neumann function [21]. The Green function exists if $J_0(k_2 R) \neq 0$.

Let us consider equation (15). Using the second Green formula

$$\int_0^R (vLu - uLv)d\rho = \int_0^R \left(v\left(\rho u'\right)' - u\left(\rho v'\right)'\right)d\rho =$$
$$= R\left(u'(R)v(R) - v'(R)u(R)\right),$$

and assuming that $v = G$, we obatin

$$\int_0^R (GLu - uLG)d\rho = R\left(u'(R-0)G(R,s) - G'(R,s)u(R-0)\right) =$$
$$= -Ru(R-0)G'(R,s),$$

since it is clear from (17) that $G(R,s) = 0$.

Using formula (15), we have

$$\int_0^R GLu_2 d\rho = \int_0^R G(\rho,s)W(\rho)d\rho.$$

Further, using (16), we obtain

$$\int_0^R u_2 LG d\rho = \int_0^R u_2(\rho)\delta(\rho - s)d\rho = u_2(s).$$

Applying all these results, from (14) we obtain the nonlinear integral equation with respect to $u_2(s)$ on interval $(0,R)$:

$$u_2(s) = \int_0^R G(\rho,s)W(\rho)d\rho + Ru_2(R-0)\left.\frac{\partial G(\rho,s)}{\partial \rho}\right|_{\rho=R}. \quad (18)$$

Then, from (13) we obtain

$$u_1(s) = \frac{\gamma}{k_2^2}\frac{\partial}{\partial s}\int_0^R G(\rho,s)W(\rho)d\rho - \frac{f_1(s)}{k_2^2} +$$
$$+ \frac{\gamma R}{k_2^2}u_2(R-0)\left.\frac{\partial^2 G(\rho,s)}{\partial \rho \partial s}\right|_{\rho=R}, \quad \text{where } \rho \le s \le R. \quad (19)$$

Notice that, multiplying functions $\mathbf{E}$, $\mathbf{H}$ by an arbitrary constant $C_0 \neq 0$ and the nonlinearity coefficient α by C_0^{-2} in (1) it does not change the Maxwell equations. This circumstance gives an opportunity to normalize the Maxwell system. Choose the normalization in the form $C = 1$ (this problem depends on the initial condition E_z^R, see the note on p. 49 for further details.). Then, from transmission conditions (10), (11) and formulas (12) we obtain

$$u_2(R-0) = K_0(k_1 R) \tag{20}$$

and

$$\varepsilon_2 u_1|_{s=R-0} + \alpha u_1 |\mathbf{u}|^2|_{s=R-0} = -\varepsilon_1 \frac{\gamma}{k_1} K_0'(k_1 R). \tag{21}$$

From formulas (12) and (21) we obtain *the dispersion equation*

$$\Delta(\gamma) \equiv \varepsilon_2 u_1(R-0) + \alpha u_1(R-0)|\mathbf{u}(R-0)|^2 + \varepsilon_1 \frac{\gamma}{k_1} K_0'(k_1 R) = 0 \tag{22}$$

under condition that functions u_1, u_2 are solution of the system

$$\begin{cases} u_1(s) = \frac{\gamma}{k_2^2} \frac{\partial}{\partial s} \int\limits_0^R G(\rho, s) W(\rho) d\rho - \frac{f_1(s)}{k_2^2} + \frac{\gamma R}{k_2^2} K_0(k_1 R) \frac{\partial^2 G}{\partial \rho \partial s}(R, s), \\ u_2(s) = \int\limits_0^R G(\rho, s) W(\rho) d\rho + R K_0(k_1 R) \frac{\partial G}{\partial \rho}(R, s) \end{cases} \tag{23}$$

(here formulas (12), (18), (19), and (20) are used).

It should be noticed that in system (23) all the functions are defined on the interval $(0, R)$ and can be found without reference to the transmission conditions and the dispersion equation. It will be shown below that under certain conditions system (23) has a unique solution and the way of its finding will be presented.

Let us transform system (23) to the more convenient form, where there are no required derivatives of functions under the integral sign. First, transform the first summands in the right-hand sides of system (23). Using integration by parts and taking into account $W = \frac{k_2^2}{k_0^2\varepsilon_2}\left(\frac{\gamma}{k_2^2}(\rho f_1)' - \rho f_2\right)$, we find

$$\int\limits_0^R G(\rho,s)(\rho f_1)' d\rho = G(\rho,s)(\rho f_1)|_0^R - \int\limits_0^R \frac{\partial G(\rho,s)}{\partial\rho}\rho f_1(\rho)d\rho =$$

$$= G(R,s)Rf_1(R) - G(0,s)\cdot 0\cdot f_1(0) - \int\limits_0^R \frac{\partial G(\rho,s)}{\partial\rho}\rho f_1(\rho)d\rho.$$

Further, we have

$$\frac{\partial}{\partial s}\int\limits_0^R G(\rho,s)(\rho f_1)' d\rho =$$

$$= \frac{\partial}{\partial s}\left(G(R,s)Rf_1(R) - \int\limits_0^R \frac{\partial G(\rho,s)}{\partial\rho}\rho f_1(\rho)d\rho\right) =$$

$$= \frac{\partial G}{\partial s}(R,s)Rf_1(R) - \frac{\partial}{\partial s}\int\limits_0^R \frac{\partial G(\rho,s)}{\partial\rho}\rho f_1(\rho)d\rho =$$

$$= -\frac{\partial}{\partial s}\int\limits_0^R \frac{\partial G(\rho,s)}{\partial\rho}\rho f_1(\rho)d\rho.$$

Then, substituting explicit expression for the Green function in this formula, we obtain

$$\frac{\partial}{\partial s}\int\limits_0^R \frac{\partial}{\partial\rho}G(\rho,s)\rho f_1(\rho)d\rho =$$

$$= \frac{\pi}{2}\frac{\partial}{\partial s}\int\limits_0^s k_2 J_0'(k_2\rho)\frac{J_0(k_2 s)N_0(k_2 R) - N_0(k_2 s)J_0(k_2 R)}{J_0(k_2 R)}\rho f_1(\rho)d\rho +$$

$$+\frac{\pi}{2}\frac{\partial}{\partial s}\int_s^R k_2 J_0(k_2 s)\frac{J_0'(k_2\rho)N_0(k_2R)-N_0'(k_2\rho)J_0(k_2R)}{J_0(k_2R)}\rho f_1(\rho)d\rho =$$

$$= \frac{\pi}{2}\frac{\partial}{\partial s}\left[\frac{k_2}{J_0(k_2R)}J_0(k_2s)N_0(k_2R)\int_0^s J_0'(k_2\rho)\rho f_1(\rho)d\rho - \right.$$

$$-\frac{k_2}{J_0(k_2R)}N_0(k_2s)J_0(k_2R)\int_0^s J_0'(k_2\rho)\rho f_1(\rho)d\rho +$$

$$+\frac{k_2}{J_0(k_2R)}J_0(k_2s)N_0(k_2R)\int_s^R J_0'(k_2\rho)\rho f_1(\rho)d\rho -$$

$$\left. -\frac{k_2}{J_0(k_2R)}J_0(k_2s)J_0(k_2R)\int_s^R N_0'(k_2\rho)\rho f_1(\rho)d\rho\right] =$$

$$= \frac{\pi}{2}k_2^2\frac{J_0'(k_2s)}{J_0(k_2R)}N_0(k_2R)\int_0^s J_0'(k_2\rho)\rho f_1(\rho)d\rho +$$

$$+\frac{\pi}{2}k_2\frac{N_0(k_2R)}{J_0(k_2R)}J_0(k_2s)J_0'(k_2s)sf_1(s) -$$

$$-\frac{\pi}{2}k_2^2\frac{J_0(k_2R)}{J_0(k_2R)}N_0'(k_2s)\int_0^s J_0'(k_2\rho)\rho f_1(\rho)d\rho -$$

$$-\frac{\pi}{2}\frac{k_2J_0(k_2R)}{J_0(k_2R)}N_0(k_2s)J_0'(k_2s)sf_1(s) +$$

$$+\frac{\pi}{2}k_2^2\frac{N_0(k_2R)}{J_0(k_2R)}J_0'(k_2s)\int_s^R k_2J_0'(\rho)\rho f_1(\rho)d\rho -$$

$$-\frac{\pi}{2}\frac{k_2N_0(k_2R)}{J_0(k_2R)}J_0(k_2s)J_0'(k_2s)sf_1(s) -$$

$$-\frac{\pi}{2}k_2^2\frac{J_0(k_2R)}{J_0(k_2R)}J_0'(k_2s)\int_s^R N_0'(k_2\rho)\rho f_1(\rho)d\rho +$$

$$+\frac{\pi}{2}k_2\frac{J_0(k_2R)}{J_0(k_2R)}J_0(k_2s)N_0'(k_2s)sf_1(s)=$$

$$=\int\limits_0^R\frac{\partial^2G}{\partial s\partial\rho}(\rho,s)\rho f_1(\rho)d\rho-$$

$$-k_2sf_1(s)\frac{\pi}{2}\left(J_0(k_2s)N_0'(k_2s)-N_0(k_2s)J_0'(k_2s)\right)=-f_1(s).$$

After transformation we obtain the system of integral equations in the final form

$$\begin{cases}u_1(s)=-\frac{\gamma^2}{k_0^2\varepsilon_2k_2^2}\int\limits_0^R\frac{\partial^2G}{\partial s\partial\rho}\rho f_1d\rho-\frac{\gamma}{k_0^2\varepsilon_2}\int\limits_0^R\frac{\partial G}{\partial s}\rho f_2d\rho-\\ \qquad\qquad-\frac{1}{k_2^2}f_1(s)+h_1(s),\\ u_2(s)=-\frac{\gamma}{k_0^2\varepsilon_2}\int\limits_0^R\frac{\partial G}{\partial\rho}\rho f_1d\rho-\frac{k_2^2}{k_0^2\varepsilon_2}\int\limits_0^R G\rho f_2d\rho+h_2(s),\end{cases}\tag{24}$$

where

$$h_1(s)=\frac{\gamma R}{k_2^2}\frac{\partial^2G(R,s)}{\partial\rho\partial s}K_0(k_1R),\tag{25}$$

$$h_2(s)=R\frac{\partial G(R,s)}{\partial\rho}K_0(k_1R).\tag{26}$$

Let us represent system (24) in the matrix operator form. Introduce the kernel matrix

$$K(\rho,s)=\{K_{nm}(\rho,s)\}_{n,m=1}^2=-\rho\begin{pmatrix}q_{11}G_{\rho s}&q_{12}G_s\\q_{21}G_\rho&q_{22}G\end{pmatrix},\tag{27}$$

where the function G indexes denote partial derivatives. Also introduce the matrix of coefficients

$$Q=\begin{pmatrix}q_{11}&q_{12}\\q_{21}&q_{22}\end{pmatrix}=\frac{1}{\varepsilon_2}\begin{pmatrix}(\gamma/k_2)^2&\gamma\\\gamma&k_2^2\end{pmatrix},\tag{28}$$

and the matrix linear integral operator

$$\mathrm{K}=\{\mathrm{K}_{nm}\}_{n,m=1}^2$$

with the operators K_{mn}, associated with system (24),

$$\mathrm{K}\mathbf{g} = \int_0^R K(\rho, s)\mathbf{g}(\rho)d\rho, \tag{29}$$

where $\mathbf{g} = (g_1, g_2)^T$.

Then, the system of integral equations can be written in the operator form

$$\mathbf{u} = \alpha\mathrm{K}(|\mathbf{u}|^2\mathbf{u}) - \alpha\mathrm{J}(|\mathbf{u}|^2\mathbf{u}) + \mathbf{h}, \tag{30}$$

where $\mathbf{h} = (h_1, h_2)^T$ and the operator J is defined by formula

$$\mathrm{J} = \frac{k_0^2}{k_2^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{31}$$

Notice that the operators K, J are linear.

Also introduce two linear operators $\mathrm{N} := \alpha(\mathrm{K} - \mathrm{J})$ and $\mathrm{N}_0 := \mathrm{K} - \mathrm{J}$.

We will consider equation (30) in $\mathbf{C}[0, R] = C[0, R] \times C[0, R]$ with the norm

$$\|\mathbf{u}\|_{\mathbf{C}}^2 = \|u_1\|_C^2 + \|u_2\|_C^2,$$

where $\|u\|_C = \max\limits_{x\in[0,R]} u(x)$.

§5. Studying of the Integral Operators Kernels

To study integral operator (29) let us consider the corresponding integral operators kernels.

Let $\Pi = (0, R) \times (0, R)$. Using properties of the Bessel and Neumann functions, let us prove that the functions $k_{11}(\rho, s)$ and $k_{22}(\rho, s)$ are continuous in (closed) square $\overline{\Pi} = [0, R] \times [0, R]$. The function $k_{12}(\rho, s)$ is bounded in $\overline{\Pi}$ and continuous in $\overline{T}^+$ and in $\overline{T}^-\backslash\{0\}$. The function $k_{21}(\rho, s)$ is bounded in $\overline{\Pi}$ and continuous in $\overline{T}^+$ and in $\overline{T}^-$, where

$$\overline{T}^+ = \{(\rho, s) \in \Pi, \rho \geq s\},\ \overline{T}^- = \{(\rho, s) \in \Pi, \rho \leq s\}.$$

By definition function $f(\rho, s)$ is continuous in $\overline{T}^{+}$ (in $\overline{T}^{-}$) if for any point $(\rho_0, s_0) \in \overline{T}^{+}$

$$\lim_{\substack{\rho\to\rho_0, s\to s_0\\(\rho,s)\in\overline{T}^{+},(\rho_0,s_0)\in\overline{T}^{+}}} f(\rho, s) = f(\rho_0, s_0)$$

or for any point $(\rho_0, s_0) \in \overline{T}^{-}$

$$\lim_{\substack{\rho\to\rho_0, s\to s_0\\(\rho,s)\in\overline{T}^{-},(\rho_0,s_0)\in\overline{T}^{-}}} f(\rho, s) = f(\rho_0, s_0).$$

The function $f(\rho, s)$ is continuous in $\overline{T}^{-}\backslash\{0\}$ if it is continuous everywhere in $\overline{T}^{-}$ (in the above meaning) except the point $\rho = 0$, $s = 0$. Under these conditions, a function $f(\rho, s)$, which is continuous in $\overline{T}^{+}$ and in $\overline{T}^{-}$, is not continuous in $\overline{\Pi}$.

In order to prove above formulated properties of the kernels it is only necessary to check the behavior of the functions $k_{11}(\rho, s)$, $k_{22}(\rho, s)$, $k_{12}(\rho, s)$, and $k_{21}(\rho, s)$ at zero, i.e. at the point $\rho = 0$, $s = 0$. Calculate the limits of the Green function and its derivatives as $\rho \to 0$, $s \to 0$. As $x \to 0$ we have

$$N_0(x) = -\frac{2}{\pi}\ln\frac{2}{\gamma x} + O(1), \qquad N_0'(x) = \frac{2}{\pi x} + O(1),$$
$$J_0(x) = 1 + O(x), \qquad J_0'(x) = -\frac{x}{2} + O(x^2).$$

Rewrite the Green function in the form

$$G(\rho, s) = \frac{\pi}{2}\frac{1}{J_0(k_2R)}\times$$
$$\times\begin{cases} J_0(k_2\rho)\left(N_0(k_2s)J_0(k_2R) - J_0(k_2s)N_0(k_2R)\right), & \rho \le s \le R,\\ J_0(k_2s)\left(N_0(k_2\rho)J_0(k_2R) - J_0(k_2\rho)N_0(k_2R)\right), & s \le \rho \le R.\end{cases}$$

Then, calculating the derivative, we obtain

$$\left.\frac{\partial G}{\partial s}\right|_{\rho\le s} = \\ = \frac{\pi}{2}\frac{J_0(k_2\rho)}{J_0(k_2R)}\left(N_0'(k_2s)J_0(k_2R)k_2 - J_0'(k_2s)N_0(k_2R)k_2\right) = \\ = \frac{\pi}{2}k_2\frac{J_0(k_2\rho)}{J_0(k_2R)}\left(N_0'(k_2s)J_0(k_2R) - J_0'(k_2s)N_0(k_2R)\right).$$

Now, calculate the derivative as $\rho\to 0,\ s\to 0$

$$\left.\frac{\partial G}{\partial s}\cdot\rho\right|_{\rho\le s} = \\ = \frac{\pi}{2}k_2\frac{J_0(k_2\rho)}{J_0(k_2R)}\left(N_0'(k_2s)J_0(k_2R) - J_0'(k_2s)N_0(k_2R)\right)\rho = \\ = \frac{\pi}{2}k_2\frac{1}{J_0(k_2R)}\left(\frac{2}{\pi k_2 s}J_0(k_2R) - \left(-\frac{k_2s}{2}\right)N_0(k_2R)\right)\rho + o(1) = \\ = \frac{1}{s}\rho + o(1),$$

where $o(1)$ denotes a function $\alpha(\rho,s)$ such that $\lim\limits_{\substack{\rho\to 0\\ s\to 0}}\alpha(\rho,s) = 0$. Since $\rho\le s$; therefore, the function is bounded in the neighborhood $\rho = 0,\ s = 0$. Notice that the limit of this function as $\rho\to 0,\ s\to 0$ does not exist. Similarly we have

$$\left.\frac{\partial G}{\partial s}\right|_{s\le\rho} = \\ = \frac{\pi}{2}k_2\frac{J_0'(k_2s)}{J_0(k_2R)}\left(N_0(k_2\rho)J_0(k_2R) - J_0(k_2\rho)N_0(k_2R)\right) = \\ = \frac{\pi}{2}k_2\frac{J_0'(k_2\rho)}{J_0(k_2R)}\left(N_0(k_2\rho)J_0(k_2R) - J_0(k_2\rho)N_0(k_2R)\right).$$

Calculating the limit, we obtain

$$\lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\partial G}{\partial s}\cdot\rho\bigg|_{s\le\rho} =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}k_2\frac{J_0'(k_2\rho)}{J_0(k_2R)}\left(N_0(k_2\rho)J_0(k_2R)-J_0(k_2\rho)N_0(k_2R)\right)\rho =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}k_2\frac{1}{J_0(k_2R)}\left(\frac{k_2s}{2}\frac{2}{\pi}\ln\frac{2}{\gamma k_2\rho}\cdot J_0(k_2R)+\frac{k_2s}{2}N_0(k_2R)\right)\rho =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}\frac{1}{J_0(k_2R)}\cdot k_2\cdot\frac{k_2s}{2}N_0(k_2R)\rho = 0.$$

Thus the function

$$\frac{\partial G}{\partial s} = -\frac{\pi}{2}\frac{k_2}{J_0(k_2R)}\times$$

$$\times\begin{cases} J_0(k_2\rho)\left(N_1(k_2s)J_0(k_2R)-J_1(k_2s)N_0(k_2R)\right), & \rho\le s,\\ J_1(k_2s)\left(N_0(k_2\rho)J_0(k_2R)-J_0(k_2\rho)N_0(k_2R)\right), & \rho\ge s,\end{cases}$$

is not continuous at zero, but it is bounded in the neighborhood of zero. Here $N_1(\rho)$ is the first-order Neumann function [21].

Further, calculate the derivative

$$\frac{\partial G}{\partial\rho}\bigg|_{\rho\le s} = \frac{\pi}{2}k_2\frac{J_0'(k_2\rho)}{J_0(k_2R)}\left(N_0(k_2s)J_0(k_2R)-J_0(k_2s)N_0(k_2R)\right).$$

Now, calculate the limit

$$\lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\partial G}{\partial\rho}\cdot\rho\bigg|_{\rho\le s} =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}k_2\frac{J_0'(k_2\rho)}{J_0(k_2R)}\left(N_0(k_2s)J_0(k_2R)-J_0(k_2s)N_0(k_2R)\right)\rho =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}\frac{k_2}{J_0(k_2R)}\left(\frac{k_2\rho}{2}\frac{2}{\pi}\ln\frac{2}{\gamma k_2 s}J_0(k_2R)+\frac{k_2\rho}{2}N_0(k_2R)\right)\rho = 0.$$

Similarly we have

$$\left.\frac{\partial G}{\partial \rho}\right|_{s\le\rho} = \frac{\pi}{2}k_2\frac{J_0(k_2 s)}{J_0(k_2 R)}\left(N_0'(k_2\rho)J_0(k_2R) - J_0'(k_2\rho)N_0(k_2R)\right).$$

Calculating the limit, we obtain

$$\lim_{\substack{\rho\to 0\\ s\to 0}} \left.\frac{\partial G}{\partial \rho}\cdot\rho\right|_{\rho\le s} =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}k_2\frac{J_0(k_2 s)}{J_0(k_2 R)}\left(N_0'(k_2\rho)J_0(k_2R) - J_0'(k_2\rho)N_0(k_2R)\right)\rho =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}\frac{k_2}{J_0(k_2 R)}\left(\frac{2}{\pi k_2\rho}\cdot J_0(k_2R) + \frac{k_2\rho}{2}N_0(k_2R)\right)\rho =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}\frac{k_2}{J_0(k_2 R)}\frac{2}{\pi k_2\rho}J_0(k_2R)\rho = 1.$$

Thus the function

$$\frac{\partial G}{\partial \rho} = -\frac{\pi}{2}\frac{k_2}{J_0(k_2R)}\times$$
$$\times\begin{cases} J_1(k_2\rho)\left(N_0(k_2s)J_0(k_2R) - J_0(k_2s)N_0(k_2R)\right), & \rho\le s,\\ J_0(k_2s)\left(N_1(k_2\rho)J_0(k_2R) - J_1(k_2\rho)N_0(k_2R)\right), & \rho\ge s,\end{cases}$$

is not continuous at zero also but it is bounded in the neighborhood of zero.

For the second derivatives, we find

$$\left.\frac{\partial^2 G}{\partial \rho\partial s}\right|_{\rho\le s} = \frac{\pi}{2}k_2^2\frac{J_0'(k_2\rho)}{J_0(k_2 R)}\left(N_0'(k_2s)J_0(k_2R) - J_0'(k_2s)N_0(k_2R)\right).$$

Calculate the limit

$$\lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\partial^2 G}{\partial\rho\partial s}\cdot\rho\bigg|_{\rho\le s} =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}k_2^2\frac{J_0'(k_2\rho)}{J_0(k_2R)}\left(N_0'(k_2s)J_0(k_2R) - J_0'(k_2s)N_0(k_2R)\right)\rho =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}k_2^2\frac{1}{J_0(k_2R)}\left(-\frac{k_2\rho}{2}\right)\left(\frac{2}{\pi k_2 s}J_0(k_2R) + \frac{k_2 s}{2}N_0(k_2R)\right)\rho =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}k_2^2\frac{1}{J_0(k_2R)}\left(-\frac{k_2\rho}{2}\right)\frac{2}{\pi k_2 s}J_0(k_2R)\rho =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{1}{J_0(k_2R)}\left(-\frac{k_2^2\rho}{2}\right)\frac{1}{r}J_0(k_2R)\rho = 0.$$

Further, similarly we obtain

$$\frac{\partial^2 G}{\partial\rho\partial s}\bigg|_{s\le\rho} = \frac{\pi}{2}k_2^2\frac{J_0'(k_2 s)}{J_0(k_2R)}\left(N_0'(k_2\rho)J_0(k_2R) - J_0'(k_2\rho)N_0(k_2R)\right).$$

Calculating the limit we obtain

$$\lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\partial^2 G}{\partial\rho\partial s}\cdot\rho\bigg|_{s\le\rho} =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}k_2^2\frac{J_0'(k_2 s)}{J_0(k_2R)}\left(N_0'(k_2\rho)J_0(k_2R) - J_0'(k_2\rho)N_0(k_2R)\right)\rho =$$

$$= \lim_{\substack{\rho\to 0\\ s\to 0}} \frac{\pi}{2}\frac{k_2^2}{J_0(k_2R)}\left(-\frac{k_2 s}{2}\right)\left(\frac{2}{\pi k_2\rho}J_0(k_2R) + \frac{k_2\rho}{2}N_0(k_2R)\right)\rho = 0.$$

The following function is continuous at zero

$$\frac{\partial^2 G}{\partial\rho\partial s} = \frac{\pi}{2}\frac{k_2^2}{J_0(k_2R)}\times$$

$$\times\begin{cases} J_1(k_2\rho)\left(N_1(k_2 s)J_0(k_2R) - J_1(k_2 s)N_0(k_2R)\right), & \rho\le s,\\ J_1(k_2 s)\left(N_1(k_2\rho)J_0(k_2R) - J_1(k_2\rho)N_0(k_2R)\right), & \rho\ge s.\end{cases}$$

Thus we proved the following proposition.

Proposition 1. *The functions* $k_{11}(\rho, s)$ *and* $k_{22}(\rho, s)$ *are continuous in* $\overline{\Pi} = [0, R] \times [0, R]$. *The function* $k_{12}(\rho, s)$ *is bounded in* $\overline{\Pi}$ *and continuous in* $\overline{T}^{+}$ *and in* $\overline{T}^{-} \setminus \{0\}$, *the function* $k_{21}(\rho, s)$ *is bounded in* $\overline{\Pi}$ *and continuous in* $\overline{T}^{+}$ *and in* $\overline{T}^{-}$.

Further, calculate the values of other functions contained in (25) and (26). We have

$$\frac{\partial G(R, s)}{\partial \rho} = \frac{\pi}{2} k_2 \frac{J_0(k_2 s)}{J_0(k_2 R)} \left(N_0'(k_2 R) J_0(k_2 R) - J_0'(k_2 R) N_0(k_2 R) \right) = \frac{\pi}{2} k_2 \frac{J_0(k_2 s)}{J_0(k_2 R)} \frac{2}{\pi k_2 R} = \frac{1}{R} \frac{J_0(k_2 s)}{J_0(k_2 R)}.$$

Similarly for the second derivative we obtain

$$\frac{\partial^2 G(R, s)}{\partial \rho \partial s} = -\frac{k_2}{R} \frac{J_1(k_2 s)}{J_0(k_2 R)}.$$

Then

$$h_1(s) = -\frac{\gamma}{k_2} \frac{J_1(k_2 s)}{J_0(k_2 R)} K_0(k_1 R), \tag{32}$$

$$h_2(s) = \frac{J_0(k_2 s)}{J_0(k_2 R)} K_0(k_1 R). \tag{33}$$

The boundedness of the operator $\mathrm{K} : \mathbf{C}[0, R] \to \mathbf{C}[0, R]$ results from the properties of the kernels. It is obvious, that the operator $\mathrm{J} : \mathbf{C}[0, R] \to \mathbf{C}[0, R]$ is bounded. The corresponding proposition with the estimations of norms of the operators will be given in the next section.

§6. ESTIMATIONS OF NORMS OF THE INTEGRAL OPERATORS

Let us estimate norms of the integral operators in $\mathbf{C}[0, R] = C[0, R] \times C[0, R]$. These estimations are required below. First, consider the scalar case. Let the integral operator be defined by formula

$$\mathrm{K}\varphi = \int_0^R K(x,y)\varphi(y)dy \tag{34}$$

with the bounded, piecewise continuous kernel $K(x,y)$ in the $[0,R]\times[0,R]$. Then

$$\left|\int_0^R K(x,y)\varphi(y)dy\right| \leq \int_0^R |K(x,y)|\,|\varphi(y)|dy \leq$$

$$\leq \max_{x\in[0,R]} |\varphi(x)| \int_0^R |K(x,y)|dy \leq \|\varphi\|_C \max_{x\in[0,R]} \int_0^R |K(x,y)|dy.$$

Therefore

$$\|K\varphi\|_C = \max_{x\in[0,R]} \left|\int_0^R K(x,y)\varphi(y)dy\right| \leq M_0 \|\varphi\|_C\,,$$

where $M_0 = \max\limits_{x\in[0,R]} \int_0^R |K(x,y)|dy$.

Hence for the norm of the operator $\mathrm{K} : C[0,R] \to C[0,R]$ we have the estimation $\|\mathrm{K}\|_{C\to C} \leq M_0$. If the kernel of the integral operator $K(x,y)$ is continuous in $[0,R]\times[0,R]$, then the equality $\|\mathrm{K}\|_{C\to C} = M_0$ holds [27]. Thus we proved the following proposition

Proposition 2. *Let* $\mathrm{K} : C[0,R] \to C[0,R]$ *be the integral operator defined by formula* (34) *with the piecewise continuous kernel* $K(x,y)$ *in* $[0,R]\times[0,R]$. *Then, the operator* K *is bounded and its norm estimation*

$$\|\mathrm{K}\|_{C\to C} \leq M_0,$$

holds, where

$$M_0 = \max_{x\in[0,R]} \int_0^R |K(x,y)|dy.$$

Let us consider the vector case. Let the matrix linear integral operator $\mathrm{K} = \{\mathrm{K}_{mn}\}_{m,n=1}^{2}$ be defined by formula

$$\mathrm{K}\varphi = \int_0^R K(x,y)\varphi(y)dy \qquad (35)$$

with the bounded kernels $K_{nm}(x,y)$. Let the kernels have the properties formulated in the Proposition 1.

Then, the following estimations

$$\begin{aligned}
\|\mathrm{K}\varphi\|_{\mathrm{C}}^2 &= \|\mathrm{K}_{11}\varphi_1 + \mathrm{K}_{12}\varphi_2\|_C^2 + \|\mathrm{K}_{21}\varphi_1 + \mathrm{K}_{22}\varphi_2\|_C^2 \le \\
&\le \left(\|\mathrm{K}_{11}\varphi_1\|_C + \|\mathrm{K}_{12}\varphi_2\|_C\right)^2 + \left(\|\mathrm{K}_{21}\varphi_1\|_C + \|\mathrm{K}_{22}\varphi_2\|_C\right)^2 \le \\
&\le \left(\|\mathrm{K}_{11}\|_{C\to C}\|\varphi_1\|_C + \|\mathrm{K}_{12}\|_{C\to C}\|\varphi_2\|_C\right)^2 + \\
&+ \left(\|\mathrm{K}_{21}\|_{C\to C}\|\varphi_1\|_C + \|\mathrm{K}_{22}\|_{C\to C}\|\varphi_2\|_C\right)^2 \le
\end{aligned}$$

$$\begin{aligned}
&\le 2\|\mathrm{K}_{11}\|_{C\to C}^2\|\varphi_1\|_C^2 + 2\|\mathrm{K}_{12}\|_{C\to C}^2\|\varphi_2\|_C^2 + \\
&+ 2\|\mathrm{K}_{21}\|_{C\to C}^2\|\varphi_1\|_C^2 + 2\|\mathrm{K}_{22}\|_{C\to C}^2\|\varphi_2\|_C^2 \le \\
&\le 2\max\left(\|\mathrm{K}_{11}\|_{C\to C}^2, \|\mathrm{K}_{12}\|_{C\to C}^2\right)\cdot\|\varphi\|_{\mathrm{C}}^2 + \\
&+ 2\max\left(\|\mathrm{K}_{21}\|_{C\to C}^2, \|\mathrm{K}_{22}\|_{C\to C}^2\right)\cdot\|\varphi\|_{\mathrm{C}}^2 = M^2\|\varphi\|_{\mathrm{C}}^2,
\end{aligned}$$

hold, where $M^2 = 2\left(\max\limits_{j=1,2}\|\mathrm{K}_{1j}\|_{C\to C}^2 + \max\limits_{j=1,2}\|\mathrm{K}_{2j}\|_{C\to C}^2\right)$.

Then $\|\mathrm{K}\|_{\mathbf{C}\to\mathbf{C}} \le M$.

Proposition 3. *Let* $\mathrm{K} : \mathbf{C}[0,R] \to \mathbf{C}[0,R]$ *be the integral operator defined by formula* (35) *with the bounded kernels* $K_{nm}(x,y)$ *in* $[0,R]\times[0,R]$, *defined by formulas* (27) *and* (28). *Then, the operator* K *is bounded and its norm estimation*

$$\|\mathrm{K}\|_{\mathbf{C}\to\mathbf{C}} \le M,$$

holds, where

$$M^2 = 2\left(\max_{j=1,2}\|\mathrm{K}_{1j}\|_{C\to C}^2 + \max_{j=1,2}\|\mathrm{K}_{2j}\|_{C\to C}^2\right).$$

§7. ITERATION METHOD FOR SOLVING INTEGRAL EQUATIONS

Approximate solutions $\mathbf{u}^n(r) = (u_1^n(r), u_2^n(r))^T$, $r \in [0, R]$ of system of integral equations (24) can be calculated with the help of the iteration process of the contraction mapping method

$$u_1^{n+1}(r) = -\frac{\alpha\gamma^2}{\varepsilon_2 k_2^2}\int\limits_0^R \frac{\partial^2 G(r,\rho)}{\partial r \partial \rho}\rho\,|\mathbf{u}^n(\rho)|^2\,u_1^n(\rho)d\rho-$$

$$-\frac{\alpha\gamma}{\varepsilon_2}\int\limits_0^R \frac{\partial G(r,\rho)}{\partial r}\rho\,|\mathbf{u}^n(\rho)|^2\,u_2^n(\rho)d\rho-\frac{\alpha k_0^2}{k_2^2}\,|\mathbf{u}^n(\rho)|^2\,u_1^n(\rho)+h_1(r),$$

$$u_2^{n+1}(r) = -\frac{\alpha\gamma}{\varepsilon_2 k_2^2}\int\limits_0^R \frac{\partial G(r,\rho)}{\partial \rho}\rho\,|\mathbf{u}^n(\rho)|^2\,u_1^n(\rho)d\rho-$$
$$-\frac{\alpha k_2^2}{\varepsilon_2}\int\limits_0^R G(r,\rho)\rho\,|\mathbf{u}^n(\rho)|^2\,u_2^n(\rho)d\rho + h_2(r). \tag{36}$$

Let us prove that the sequence $u_1^n(r), u_2^n(r)$ converges uniformly to the solution of system (24) in regard the right-hand side of system (24) defines the contracting operator. Below indexes of norms of operators are omitted since it is clear from the context what case (scalar or vector) is considered.

Theorem 1. *Let* $B_{r_0} \equiv \{\mathrm{u} : \|\mathrm{u}\| \le r_0\}$ *be the ball of radius* r_0 *with centre at zero. Also let two conditions*

$$q := 3ar_0^2\,\|\mathrm{K} - \mathrm{J}\| < 1, \tag{37}$$
$$\alpha r_0^3\,\|\mathrm{K} - \mathrm{J}\| + \|\mathbf{h}\| \le r_0 \tag{38}$$

hold. Then, the unique solution $\mathbf{u} \in B_{r_0}$ *of equation* (*or system* (24)) (30) *exists. The sequence of approximate solutions* $\mathbf{u}^n \in B_{r_0}$ *of equation* (30) (*or system* (24)) *defined by the iteration process*

$$\mathbf{u}^{n+1} = \alpha\mathrm{K}\left(|\mathbf{u}^n|^2\mathbf{u}^n\right) - \alpha\mathrm{J}\left(|\mathbf{u}^n|^2\mathbf{u}^n\right) + \mathbf{h}$$

(or (36)), converges in $\mathbf{C}[0,R]$ *to the (unique) exact solution* $\mathbf{u} \in B_{r_0}$ *of equation (30) (or system (24)) for any initial approximation* $\mathbf{u}^0 \in B_{r_0}$ *with the geometric progression rate* q.

Proof. Let us consider the equation $\mathbf{u} = A(\mathbf{u})$ with the nonlinear operator

$$\mathrm{A}(\mathbf{u}) \equiv \alpha \mathrm{K}\left(|\mathbf{u}|^2\mathbf{u}\right) - \alpha \mathrm{J}\left(|\mathbf{u}|^2\mathbf{u}\right) + \mathbf{h}$$

in $\mathbf{C}[0,R]$, where $\mathbf{h}$ is defined by formulas (32), (33).

Let $\mathbf{u}$, $\mathbf{v} \in B_{r_0}$; $\|\mathbf{u}\| \le r_0$, $\|\mathbf{v}\| \le r_0$, then

$$\|\mathrm{A}(\mathbf{u}) - \mathrm{A}(\mathbf{v})\| = \alpha \left\|\mathrm{K}(|\mathbf{u}|^2\,\mathbf{u} - |\mathbf{v}|^2\,\mathbf{v}) - \mathrm{J}(|\mathbf{u}|^2\,\mathbf{u} - |\mathbf{v}|^2\,\mathbf{v})\right\| \le \\ \le 3\alpha\,\|\mathrm{K} - \mathrm{J}\|\, r_0^2\,\|\mathbf{u} - \mathbf{v}\|\,. \quad (39)$$

Let us prove estimation (39). Indeed,

$$\left\||\mathbf{u}|^2\mathbf{u} - |\mathbf{v}|^2\mathbf{v}\right\| = \left\|\left(|\mathbf{u}|^2\mathbf{u} - |\mathbf{v}|^2\mathbf{u}\right) + \left(|\mathbf{v}|^2\mathbf{u} - |\mathbf{v}|^2\mathbf{v}\right)\right\| \le \\ \le \left\|\left(|\mathbf{u}|^2\mathbf{u} - |\mathbf{v}|^2\mathbf{u}\right)\right\| + \left\|\left(|\mathbf{v}|^2\mathbf{u} - |\mathbf{v}|^2\mathbf{v}\right)\right\| \le \\ \le \left\|\left(|\mathbf{u}|^2 - |\mathbf{v}|^2\right)\right\|\|\mathbf{u}\| + \left\|\left(|\mathbf{v}|^2\right)\right\|\|\mathbf{u} - \mathbf{v}\| = \\ = \|(|\mathbf{u}| - |\mathbf{v}|)\|\,\|(|\mathbf{u}| + |\mathbf{v}|)\|\,\|\mathbf{u}\| + \|\mathbf{v}\|^2\,\|\mathbf{u} - \mathbf{v}\| \le \\ \le \|(|\mathbf{u}| - |\mathbf{v}|)\|\,(\|\mathbf{u}\| + \|\mathbf{v}\|)\,\|\mathbf{u}\| + \|\mathbf{v}\|^2\,\|\mathbf{u} - \mathbf{v}\|\,.$$

Taking into account

$$|\mathbf{u}| \le |\mathbf{u} - \mathbf{v}| + |\mathbf{v}|,\ |\mathbf{u}| - |\mathbf{v}| \le |\mathbf{u} - \mathbf{v}|$$

and, similarly,

$$|\mathbf{v}| \le |\mathbf{u} - \mathbf{v}| + |\mathbf{u}|,\ |\mathbf{v}| - |\mathbf{u}| \le |\mathbf{u} - \mathbf{v}|,$$

we obtain

$$|(|\mathbf{u}| - |\mathbf{v}|)| \le |\mathbf{u} - \mathbf{v}| \le |\mathbf{u} - \mathbf{v}\|,$$

therefore

$$\|(|\mathbf{u}| - |\mathbf{v}|)\| \le \|\mathbf{u} - \mathbf{v}\|.$$

Then

$$\|(|\mathbf{u}|-|\mathbf{v}|)\| \left(\|\mathbf{u}\|+\|\mathbf{v}\|\right) \|\mathbf{u}\| + \|\mathbf{v}\|^2 \|\mathbf{u}-\mathbf{v}\| \leq$$
$$\leq \|\mathbf{u}-\mathbf{v}\| \left(\|\mathbf{u}\|+\|\mathbf{v}\|\right) \|\mathbf{u}\| + \|\mathbf{v}\|^2 \|\mathbf{u}-\mathbf{v}\| \leq$$
$$\leq \left(2r_0^2 + r_0^2\right) \|\mathbf{u}-\mathbf{v}\| = 3r_0^2 \|\mathbf{u}-\mathbf{v}\|.$$

We obtain

$$\left\| |\mathbf{u}|^2\mathbf{u} - |\mathbf{v}|^2\mathbf{v} \right\| \leq 3r_0^2 \|\mathbf{u}-\mathbf{v}\|. \tag{40}$$

Estimation (39) follows from above estimations. We can see that

$$\|\mathrm{A}(\mathbf{u})\| = \left\| \alpha\mathrm{K}\left(|\mathbf{u}|^2\mathbf{u}\right) - \alpha\mathrm{J}\left(|\mathbf{u}|^2\mathbf{u}\right) + \mathbf{h} \right\| \leq \alpha r_0^3 \|\mathrm{K}-\mathrm{J}\| + \|\mathbf{h}\|.$$

If condition (38) holds, then operator A maps the ball B_{r_0} into itself. From estimations (37) and (38) it follows that the operator A is contracting in the ball B_{r_0}. The principle of contracting mappings [67] implies all statements of the theorem. The theorem is proved.

Choosing sufficiently great radius of the ball r_0 in order that the estimation $\|\mathbf{h}\| < r_0$ holds and then choosing sufficiently small α it is easy to see that estimations (37) and (38) are satisfied.

Let us consider condition (38) in detail. The following consideration requires the auxiliary number cubic equation

$$\|\mathrm{N}\| r_0^3 + \|\mathbf{h}\| = r_0, \tag{41}$$

where the operator norm $\|\mathrm{N}\| = \alpha\|\mathrm{K}-\mathrm{J}\| > 0$.

Let us consider the equation

$$r_0 - \|\mathrm{N}\| r_0^3 = \|\mathbf{h}\| \tag{42}$$

and the function $y(r_0) := r_0 - \|\mathrm{N}\| r_0^3$.

It is easy to show that the function $y(r_0)$ has only one positive maximum point $r_{\max} = \frac{1}{\sqrt{3\|\mathrm{N}\|}}$. The value of the function at this point is $y_{\max} = y\left(r_{\max}\right) = \frac{2}{3\sqrt{3\|\mathrm{N}\|}}$.

Then under condition $0 \le \|\mathbf{h}\| < \frac{2}{3}\frac{1}{\sqrt{3\|\mathrm{N}\|}}$ equation (42) has two nonnegative roots r_* and r^*, $r_* \le r^*$. The roots satisfy the inequalities

$$0 \le r_* \le \frac{1}{\sqrt{3\|\mathrm{N}\|}}; \ \frac{1}{\sqrt{3\|\mathrm{N}\|}} \le r^* \le \frac{1}{\sqrt{\|\mathrm{N}\|}}.$$

These roots can be written as solutions of the following cubic equation

$$r_0^3 - \frac{1}{\sqrt{\|\mathrm{N}\|}} r_0 + \frac{\|\mathbf{h}\|}{\|\mathrm{N}\|} = 0.$$

We have

$$r_* = -\frac{2}{\sqrt{3\|\mathrm{N}\|}} \cos\left(\frac{\arccos\left(\frac{3\sqrt{3}}{2}\|\mathbf{h}\|\sqrt{\|\mathrm{N}\|}\right)}{3} - \frac{2\pi}{3}\right), \tag{43}$$

$$r^* = -\frac{2}{\sqrt{3\|\mathrm{N}\|}} \cos\left(\frac{\arccos\left(\frac{3\sqrt{3}}{2}\|\mathbf{h}\|\sqrt{\|\mathrm{N}\|}\right)}{3} + \frac{2\pi}{3}\right). \tag{44}$$

If $\|\mathbf{h}\| = 0$, then $r_* = 0$ and $r^* = \frac{1}{\sqrt{\|\mathrm{N}\|}}$.

If $0 < \|\mathbf{h}\| < \frac{2}{3}\frac{1}{\sqrt{3\|\mathrm{N}\|}}$, then

$$r_* < \frac{1}{\sqrt{3\|\mathrm{N}\|}}. \tag{45}$$

If $\|\mathbf{h}\| = \frac{2}{3}\frac{1}{\sqrt{3\|\mathrm{N}\|}}$, then $r_* = r^* = \frac{2}{3}\frac{1}{\sqrt{3\|\mathrm{N}\|}}$.

Thus, we proved the following lemma.

Lemma 1. *If the inequality*

$$0 \le \|\mathbf{h}\| < \frac{2}{3}\frac{1}{\sqrt{3\|\mathrm{N}\|}}, \tag{46}$$

holds, then equation (41) *has two nonnegative roots* r_* *and* r^*; *and* $r_* < r^*$.

Let us prove that if condition (46) holds, then equation (30) has a unique solution in the ball $B_{r_*} \equiv \{\mathbf{u} : \|\mathbf{u}\| \le r_*\}$.

Theorem 2. *If* $\alpha \le A^2$, *where*

$$A = \frac{2}{3}\frac{1}{\|\mathbf{h}\|\sqrt{3\|\mathrm{N}_0\|}}$$

and $\|\mathrm{N}_0\| := \|\mathrm{K}-\mathrm{J}\|(>0)$, *then equation* (30) *has a unique solution* $\mathbf{u}$ *in the ball* $B_{r_*} \equiv \{\mathbf{u} : \|\mathbf{u}\| \le r_*\}$ *and* $\mathbf{u} \in \mathbf{C}[0,R]$, $\|\mathbf{u}\| \le r_*$.

Proof. If $\mathbf{u} \in B_{r_*}$, then

$$\|\mathrm{A}(\mathbf{u})\| = \left\|\alpha \mathrm{K}\left(|\mathbf{u}|^2\mathbf{u}\right) - \alpha \mathrm{J}\left(|\mathbf{u}|^2\mathbf{u}\right) + \mathbf{h}\right\| \le \alpha r_*^3\|\mathrm{K}-\mathrm{J}\| + \|\mathbf{h}\| = r_*.$$

If $\mathbf{u}$, $\mathbf{v} \in B_{r_*}$, then

$$\begin{aligned}\|\mathrm{A}(\mathbf{u}) - \mathrm{A}(\mathbf{v})\| &= \\ &= \alpha\left\|\mathrm{K}\left(|\mathbf{u}|^2\mathbf{u} - |\mathbf{v}|^2\mathbf{v}\right) - \mathrm{J}\left(|\mathbf{u}|^2\mathbf{u} - |\mathbf{v}|^2\mathbf{v}\right)\right\| \le \\ &\le 3\alpha\|\mathrm{K}-\mathrm{J}\|r_*^2\|\mathbf{u}-\mathbf{v}\|.\end{aligned}$$

Since $\alpha \le A^2$; therefore, the vector $\mathbf{h}$ satisfies condition (46). Thus equation (45) holds. And we obtain that

$$q = 3\alpha r_*^2\|\mathrm{K}-\mathrm{J}\| = 3\|\mathrm{N}\|r_*^2 < 1.$$

Consequently, both inequalities (37) and (38) hold.

Thus A maps B_{r_*} into itself and is a contracting operator in B_{r_*}. Therefore equation (30) has a unique solution in the ball B_{r_*}. The theorem is proved.

It should be noticed that $A > 0$ does not depend on α.

The results about the properties of the boundary value problem solutions will be proved in the next sections under certain sufficient conditions for the problem's parameters. Especially the proposition about eigenvalues (solutions of dispersion equation (22)) existence for the nonlinear boundary eigenvalue problem will be proved. The small parameter method will be used for proving. In this problem the nonlinearity coefficient α is the small parameter. This is the natural approach, as it is known [4], that Kerr law (which we use in this work) holds for small α.

§8. Theorem of Continuity Dependence of the Solution on the Spectral Parameter

The proposition on the parameter continuity dependence of the solutions of integral equation (30) will be needed below. Rewrite equation (30) in the following form

$$\mathbf{u} = \mathrm{N}\left(|\mathbf{u}|^2\mathbf{u}\right) + \mathbf{h},$$

where the operator

$$\mathrm{N} := \alpha(\mathrm{K} - \mathrm{J})$$

with the matrix kernels

$$N(\rho, s) := \alpha(K(\rho, s) - J(\rho, s))$$

is defined by formulas (24)–(31).

Theorem 3. *Let the matrix operator kernels* N *and right-hand side* $\mathbf{h}$ *of equation* (30) *continuously depend on the parameter* $\gamma \in \Gamma_0$, $\mathrm{N}(\gamma) \subset \mathbf{C}(\Gamma_0)$, $\mathbf{h}(\gamma) \subset \mathbf{C}(\Gamma_0)$, *on certain real segment* Γ_0. *Let also*

$$\|\mathbf{h}(\gamma)\| \leq \frac{2}{3}\frac{1}{\sqrt{3\|\mathrm{N}(\gamma)\|}}. \tag{47}$$

Then the solutions $\mathbf{u}(\gamma)$ *of equation* (30) *for* $\gamma \in \Gamma_0$ *exist, are unique and continuously depend on the parameter* γ, $\mathbf{u}(\gamma) \subset \mathbf{C}(\Gamma_0)$.

Proof. Consider equation (30). Under the assumptions of the theorem, the existence and uniqueness of solutions $\mathbf{u}(\gamma)$ follow from Theorem 2. Let us prove that these solutions depend continuously on the spectral parameter γ.

It is easy to see from the formula (43) that $r_*(\gamma)$ continuously depends on γ on segment Γ_0. Let $r_{**} = \max\limits_{\gamma\in\Gamma_0} r_*(\gamma)$ and maximum be achieved at a point γ_*, $r_*(\gamma_*) = r_{**}$. Choose $\gamma + \Delta\gamma \in \Gamma_0$, then $r_*(\gamma) \leq r_{**}$ and $r_*(\gamma + \Delta\gamma) \leq r_{**}$.

Furthermore, let $Q_0 = \max\limits_{\gamma\in\Gamma_0}(3r_*^2(\gamma)\|\mathrm{N}(\gamma)\|)$ and maximum be achieved at a point $\widehat{\gamma} \in \Gamma_0$, $Q_0 = 3r_*^2(\widehat{\gamma})\,\|\mathrm{N}(\widehat{\gamma})\|$. Then $Q_0 < 1$ by virtue of assumption (47) of the theorem.

First, let us assume that

$$\|\mathbf{u}(\gamma)\| \geq \|\mathbf{u}(\gamma+\Delta\gamma)\|. \tag{48}$$

Then the following inequalities are valid:

$$|\mathbf{u}(s,\gamma+\Delta\gamma)-\mathbf{u}(s,\gamma)| =$$

$$= \left| \int_0^R N(\gamma+\Delta\gamma,\rho,s)|\mathbf{u}(\rho,\gamma+\Delta\gamma)|^2\mathbf{u}(\rho,\gamma+\Delta\gamma)d\rho - \right.$$

$$\left. - \int_0^R N(\gamma,\rho,s)|\mathbf{u}(\rho,\gamma)|^2\mathbf{u}(\rho,\gamma)d\rho + \mathbf{h}(s,\gamma+\Delta\gamma) - \mathbf{h}(s,\gamma) \right| \leq$$

$$\leq \left| \int_0^R \left(N(\gamma+\Delta\gamma,\rho,s) - N(\gamma,\rho,s)\right)|\mathbf{u}(\rho,\gamma+\Delta\gamma)|^2\mathbf{u}(\rho,\gamma+\Delta\gamma)d\rho + \right.$$

$$\left. + \int_0^R N(\gamma,\rho,s)\left(|\mathbf{u}(\rho,\gamma+\Delta\gamma)|^2\mathbf{u}(\rho,\gamma+\Delta\gamma) - |\mathbf{u}(\rho,\gamma)|^2\mathbf{u}(\rho,\gamma)\right)d\rho \right| +$$

$$+ |\mathbf{h}(s,\gamma+\Delta\gamma)-\mathbf{h}(s,\gamma)|,$$

therefore (see the proof of Theorem 2)

$$\|\mathbf{u}(\gamma+\Delta\gamma)-\mathbf{u}(\gamma)\| \leq$$

$$\leq r_*^3(\gamma)\|N(\gamma+\Delta\gamma)-N(\gamma)\| +$$

$$+ \|\mathbf{u}(\gamma+\Delta\gamma)-\mathbf{u}(\gamma)\|3r_*^2(\gamma)\|N(\gamma)\| + \|\mathbf{h}(\gamma+\Delta\gamma)-\mathbf{h}(\gamma)\|.$$

Condition (48) is used above.

Then, it follows that

$$\|\mathbf{u}(\gamma+\Delta\gamma)-\mathbf{u}(\gamma)\| \leq$$

$$\leq \frac{r_*^3(\gamma)\|N(\gamma+\Delta\gamma)-N(\gamma)\| + \|\mathbf{h}(\gamma+\Delta\gamma)-\mathbf{h}(\gamma)\|}{1-3r_*^2(\gamma)\|N(\gamma)\|}$$

and

$$\|\mathbf{u}(\gamma+\Delta\gamma)-\mathbf{u}(\gamma)\| \leq \\ \leq \frac{r_{**}^3\|N(\gamma+\Delta\gamma)-N(\gamma)\|+\|\mathbf{h}(\gamma+\Delta\gamma)-\mathbf{h}(\gamma)\|}{1-Q_0}, \quad (49)$$

where Q_0 and r_{**} do not depend on γ.

Now, let $\|\mathbf{u}(\gamma)\| \leq \|\mathbf{u}(\gamma+\Delta\gamma)\|$. Then, all the preceding estimates remain valid if we replace γ by $\gamma+\Delta\gamma$ and $\gamma+\Delta\gamma$ by γ. Thus, estimate (49) also remains valid. The theorem is proved.

§9. Theorems of Existence and Uniqueness

Rewrite dispersion equation (22) in the following form

$$\Delta(\gamma) \equiv \varepsilon_2 u_1(R-0) + \alpha u_1(R-0)|\mathbf{u}(R-0)|^2 + \varepsilon_1 \frac{\gamma}{k_1} K_0'(k_1 R) = 0,$$

where $u_1(R-0)$ is defined from the first equation of system (24).

Using formula $J_1(z)N_0(z) - J_0(z)N_1(z) = \frac{2}{\pi z}$ and formula (17) for the Green function, it is easy to show that the following formulas $\frac{\partial G}{\partial s}\Big|_{s=R-0} = \frac{1}{R}\frac{J_0(k_2\rho)}{J_0(k_2R)}$, $\frac{\partial^2 G}{\partial\rho\partial s}\Big|_{s=R-0} = -\frac{k_2}{R}\frac{J_1(k_2\rho)}{J_0(k_2R)}$ are valid.

Now, using the above results and the first equation of system (24), we find

$$u_1(R-0) = \frac{\gamma^2}{k_2 k_0^2 \varepsilon_2 R}\frac{1}{J_0(k_2R)}\int_0^R \rho J_1(k_2\rho) f_1 d\rho - \\ - \frac{\gamma}{k_0^2\varepsilon_2 R}\frac{1}{J_0(k_2R)}\int_0^R \rho J_0(k_2\rho) f_2 d\rho - \\ - \frac{1}{k_2^2} f_1(R-0) - \frac{\gamma}{k_2}\frac{J_1(k_2R)}{J_0(k_2R)} K_0(k_1R).$$

(we keep in mind that $f_1 = \alpha k_0^2 |\mathbf{u}|^2 u_1$ and $f_2 = \alpha k_0^2 |\mathbf{u}|^2 u_2$).

Put all the terms without α together in the left side and others put in the right side. We obtain

$$\varepsilon_2\frac{\gamma}{k_2}\frac{J_1(k_2R)}{J_0(k_2R)}K_0(k_1R)-\varepsilon_1\frac{\gamma}{k_1}K_0'(k_1R)=\alpha\widetilde{F}(\gamma), \tag{50}$$

where

$$\widetilde{F}(\gamma)=\frac{\gamma^2}{k_2R}\frac{1}{J_0(k_2R)}\int\limits_0^R\rho J_1(k_2\rho)|\mathbf{u}|^2u_1d\rho-$$
$$-\frac{\gamma}{R}\frac{1}{J_0(k_2R)}\int\limits_0^R\rho J_0(k_2\rho)|\mathbf{u}|^2u_2d\rho-$$
$$-\varepsilon_2\frac{k_0^2}{k_2^2}|\mathbf{u}(R-0)|^2u_1(R-0)+|\mathbf{u}(R-0)|^2u_1(R-0). \tag{51}$$

Multiplying equation (50) by $\frac{k_1k_2}{\gamma}J_0(k_2R)$ and taking into account equalities $k_2^2=k_0^2\varepsilon_2-\gamma^2$ and $K_0'(z)=-K_1(z)$, we obtain

$$\varepsilon_2k_1J_1(k_1R)K_0(k_1R)+\varepsilon_1k_2J_0(k_2R)K_1(k_1R)=\alpha F(\gamma), \tag{52}$$

where

$$F(\gamma)=\frac{k_1}{R}\int\limits_0^R\rho\left(\gamma J_1(k_2\rho)u_1(\rho)-k_2J_0(k_2\rho)u_2(\rho)\right)|\mathbf{u}|^2d\rho-$$
$$-\gamma\frac{k_1}{k_2}J_0(k_2R)|\mathbf{u}(R-0)|^2u_1(R-0). \tag{53}$$

A solution of integral equation system (24) depends on α. Function (53) is expressed in terms of this solution therefore this function implicitly depends on the nonlinearity coefficient α. Nevertheless, this function can be evaluated by a constant (in a certain ball) and this constant does not depend on α. It allows us to make right-hand side (52) sufficiently small for sufficiently small α. The meaning of the above transformations is in the following. We consider equation (52) and system (24) as equations with small parameters. It is possible to consider the equations in this way since the nonlinearity coefficient α in Kerr law is small (it follows from physical consideration).

Let us consider the left-hand side of equation (52). It corresponds to the dispersion equation for a linear medium inside the waveguide, i.e. for $\alpha = 0$ (see [34, 64])

$$g(\gamma) \equiv \varepsilon_2 k_1 J_1(k_1 R) K_0(k_1 R) + \varepsilon_1 k_2 J_0(k_2 R) K_1(k_1 R) = 0.$$

Introduce the notation $\lambda_{1m} := k_0^2\varepsilon_2 - \frac{j_{1m}^2}{R^2}$, $\lambda_{2m} := k_0^2\varepsilon_2 - \frac{j_{0m}^2}{R^2}$, where j_{0m} is the m-th positive root of the equation $J_0(x) = 0$ and j_{1m} is the m-th positive root of the equation $J_1(x) = 0$; $m = 1, 2, ...$

It is known that $j_{01} < j_{11} < j_{02} < j_{12} < j_{03} < j_{13} < ...$ [21]. Then $\lambda_{21} > \lambda_{11} > \lambda_{22} > \lambda_{12} > \lambda_{23} > \lambda_{13} > ...$

It is obvious that

$$\operatorname{sign} J_1\left(R\sqrt{k_0^2\varepsilon_2 - \lambda_{2m}}\right) = \operatorname{sign} J_1(j_{0m}) = (-1)^{m+1},$$
$$\operatorname{sign} J_0\left(R\sqrt{k_0^2\varepsilon_2 - \lambda_{1m}}\right) = \operatorname{sign} J_0(j_{1m}) = (-1)^{m}.$$

The above formulas implies (taking into account that functions $K_0(x)$ and $K_1(x)$ are positive for $x > 0$)

$$\operatorname{sign} g\left(\sqrt{\lambda_{1m}}\right) = (-1)^m,\ \operatorname{sign} g\left(\sqrt{\lambda_{2m}}\right) = (-1)^{m+1}.$$

Thus there is at least one root γ_{0i} of equation $g(\gamma) = 0$ on interval $\left(\sqrt{\lambda_{1i}}, \sqrt{\lambda_{2i}}\right)$ if $k_0^2\varepsilon_1 < \lambda_{1i}$ and $\lambda_{2i} < k_0^2\varepsilon_2$, i.e. $g(\gamma_{0i}) = 0$ for $\gamma_{0i} \in \left(\sqrt{\lambda_{1i}}, \sqrt{\lambda_{2i}}\right)$.

Before the proving of the existence eigenvalues theorem for the nonlinear boundary problem P, it should be noticed that the points $\sqrt{\lambda_{2i}}$ are poles of Green's function (17). The Green function is not defined at these points. Therefore, we can choose sufficiently small numbers $\delta_i > 0$ such that the conditions

$$\operatorname{sign} g\left(\sqrt{\lambda_{2i}} - \delta_i\right) = (-1)^{i+1}, \tag{54}$$

and

$$\sqrt{\lambda_{2i}} - \delta_i > \gamma_{0i} \tag{55}$$

are fulfilled.

Let us form the segments $\Gamma_i := \left[\sqrt{\lambda_{1i}}, \sqrt{\lambda_{2i}} - \delta_i\right]$. The function $g(\gamma)$ has different signs at the different extremities of the intervals Γ_i and vanishes at the point $\gamma_{0i} \in \left(\sqrt{\lambda_{1i}}, \sqrt{\lambda_{2i}} - \delta_i\right)$ under conditions (54) and (55). Let $\lambda_{1m} > k_0^2\varepsilon_1$ for certain $m \geq 1$. Denote by $\Gamma := \bigcup\limits_{i=1}^{m} \Gamma_i$. Then we obtain

Theorem 4. *Let the numbers $\varepsilon_1, \varepsilon_2, \alpha$ satisfy the conditions $\varepsilon_2 > \varepsilon_1 > 0$, $0 < \alpha \leq \alpha_0$, where*

$$\alpha_0 = \min\left(\min_{\gamma\in\Gamma} A^2(\gamma), \frac{\min\limits_{1\leq l\leq 2, 1\leq i\leq m} \left|g(\sqrt{\lambda_{li}})\right|}{0.3R^2\left(\max\limits_{\gamma\in\Gamma} r_*(\gamma)\right)^3}\right), \tag{56}$$

$$A(\gamma) = \frac{2}{3}\frac{1}{\|\mathbf{h}(\gamma)}\|\sqrt{3\|\mathrm{N}_0(\gamma)\|}\|,$$

and the condition

$$\lambda_{1m} > k_0^2\varepsilon_1 \tag{57}$$

holds for certain $m \geq 1$. Then, there are at least m values $\gamma_i, i = 1, ..., m, \sqrt{\lambda_{1i}} < \gamma_i < \sqrt{\lambda_{2i}} - \delta_i$ such that the problem P has a nontrivial solution.

Proof. The Green function exists for all $\gamma \in \Gamma$ by virtue of choosing the values $\delta_i > 0$ $(i \geq 1)$ (see conditions (54) and (55)). It follows from the kernels and the right-hand sides of matrix integral operator that $A = A(\gamma)$ is a continuous function on the segment $\gamma \in \Gamma$. Let $A_1 = \min\limits_{\gamma\in\Gamma} A(\gamma)$ and choose $\alpha < A_1^2$. In accordance with Theorem 2 the unique solution $u = u(\gamma)$ of system (24) exists for each $\gamma \in \Gamma$. This solution is a continuous function and $\|u\| \leq r_* = r_*(\gamma)$. Let $r_{00} = \max\limits_{\gamma\in\Gamma} r_*(\gamma)$. Evaluating function (53), we obtain

$$|F(\gamma, R; u)| \leq Cr_{00}^3.$$

The function $g(\gamma)$ is continuous and the equation $g(\gamma) = 0$ has the root γ_{0i} inside the segment Γ_i, $\sqrt{\lambda_{1i}} < \gamma_{0i} < \sqrt{\lambda_{2i}}$. Let us denote

$$M_1 = \min_{1 \le i \le m} \left| g\left(\sqrt{\lambda_{1i}}\right) \right|, \quad M_2 = \min_{1 \le i \le m} \left| g\left(\sqrt{\lambda_{2i}} - \delta_i\right) \right|.$$

Then the value $\widetilde{M} = \min\{M_1, M_2\}$ is positive ($\widetilde{M} > 0$) and does not depend on the parameter α.

If $\alpha \le \frac{\widetilde{M}}{C r_{00}^3}$, then

$$(g(\lambda_{1i}) - \alpha F(\lambda_{1i})) \left(g\left(\sqrt{\lambda_{2i}} - \delta_i\right) - \alpha F\left(\sqrt{\lambda_{2i}} - \delta_i\right)\right) < 0.$$

Since $g(\gamma) - \alpha F(\gamma, R; u)$ is also a continuous function; therefore, the equation $g(\gamma) - \alpha F(\gamma, R; u) = 0$ has the root γ_i inside Γ_i, $\sqrt{\lambda_{1i}} < \gamma_i < \sqrt{\lambda_{2i}} - \delta_i$. We can choose $\alpha_0 = \min\left\{A_1^2, \frac{\widetilde{M}}{C r_{00}^3}\right\}$. The theorem is proved.

From Theorem 4, it follows that, under the above assumptions, there exist axially symmetrical propagating TM waves in cylindrical dielectric waveguides of circular cross-section filled with a nonmagnetic isotropic medium with Kerr nonlinearity. This result generalizes the well-known similar statement for dielectric waveguides of circular cross-section filled with a linear medium (i.e., $\alpha = 0$) [40].

From the condition $\lambda_{1m} > k_0^2 \varepsilon_1$ it follows that $R^2 > \frac{j_{11}^2}{(\varepsilon_2 - \varepsilon_1) k_0^2}$. Thus radius R can not be arbitrary small (similarly with the existence of the cut-off radius in a linear case). Taking into account this fact it is easy to see that the sufficient conditions for the nontrivial solution existence of the problem depend on not only the nonlinearity coefficient α smallness but on the waveguide radius R and parameter ε_2 also.

§10. Iteration Method and Estimation of the Convergence Rate

Approximate solutions $\mathbf{u}^n(s) = (u_1^n(s), u_2^n(s))^T$ of integral equations system (24) can be calculated by means of the iteration process

$$\mathbf{u}^{n+1} = \alpha(\mathrm{K} - \mathrm{J}) \left(|\mathbf{u}^n|^2 \mathbf{u}^n \right) + \mathbf{h}. \tag{58}$$

As it is proved in Theorem 1, the sequence $\mathbf{u}(s)$ converges uniformly to the solution $\mathbf{u}(s) = (u_1(s), u_2(s))^T$ of equation (24). And the rate of convergence of the iteration process is also known [67]. Particularly, if the initial approximation is $\mathbf{u}^0(s) = (0,0)^T$, then we obtain the following estimation of the iteration process convergence rate.

Proposition 4. *Let* $\mathbf{u}^0 = (0,0)^T$. *The sequence of approximate solutions* $\mathbf{u}^n = (u_1^n, u_2^n)^T$ *of system* (24), *defined by means of iteration process* (58), *exists and converges in the norm* $\mathbf{C}[0,R]$ *to the (unique) exact solution* $\mathbf{u}$ *of system* (24) *and the convergence rate estimation*

$$\|\mathbf{u} - \mathbf{u}^n\| \leq \frac{q^n}{1-q}\|\mathbf{h}\|, n \to \infty,$$

is valid, where $q := 3\alpha r_*^2 \|\mathrm{K} - \mathrm{J}\| < 1$ *is the coefficient of contraction of the mapping.*

§11. Convergence Theorem of the Iteration Method

Let us formulate the iteration method to calculate approximate eigenvalues of the boundary problem P. Also we prove the existence and convergence (the approximate solution to the exact one) theorems.

Theorem 5. *Suppose that* $\varepsilon_2 > \varepsilon_1 > 0$ and $0 < \alpha \leq \alpha_0$, *where* α_0 *is defined by* (56), *and condition* (57) *holds for certain* $m \geq 1$. *Then, for each* $n \geq 0$ *at least* m *values* $\gamma_i^{(n)}, i = 1, ..., m$, *exist. These* $\gamma_i^{(n)}$ *satisfy the inequality* $\sqrt{\lambda_{1i}} < \gamma_i^{(n)} < \sqrt{\lambda_{2i}} - \delta_i$ *and are roots of the equation*

$$k_1^{(n)} \varepsilon_2 K_1\left(k_1^{(n)} R\right) J_0\left(k_2^{(n)} R\right) + k_2^{(n)} \varepsilon_1 K_0\left(k_1^{(n)} R\right) J_1\left(k_2^{(n)} R\right) = \\ = \alpha F(\gamma^{(n)}), \quad (59)$$

where $k_1^{(n)} = \sqrt{\left(\gamma^{(n)}\right)^2 - \varepsilon_1}$, $k_2^{(n)} = \sqrt{\varepsilon_2 - \left(\gamma^{(n)}\right)^2}$ *and* $\mathbf{u}^n$ *is defined by* (58).

Proof. Since (58) is valid, for each $n \geq 0$ functions $\mathbf{u}^n$ are continuous. Thus it is enough to repeat the proof of Theorem 4 with changing $\mathbf{u}$ by $\mathbf{u}^n$ and check the conditions $\|\mathbf{u}^n\| \leq r_* = r_*(\gamma)$. This inequality is fulfilled since all the iterations $\mathbf{u}^n$ lie inside the ball B_{r_*} [67] if an initial approximation lies inside the ball B_{r_*} (which holds).

Theorem 5 states the existence of approximate eigenvalues of the boundary problem P. Equation (59) is the approximate dispersion equation for the boundary problem P. This equation uses the (known) vector $\mathbf{u}^n$ instead of (unknown) $\mathbf{u}$. It is the only distinction between this equation and the exact dispersion equation.

The following theorem states the convergence of approximate eigenvalues to the exact ones.

Theorem 6. *Let* $\varepsilon_1, \varepsilon_2, a$, *satisfy the condition* $\varepsilon_2 > \varepsilon_1 > 0$, $0 < \alpha \leq \alpha_0$, *where* α_0 *is defined by* (56), *and condition* (57) *holds for certain* $m \geq 1$. *Let* γ_i *and* $\gamma_i^{(n)}$ *be exact and approximate eigenvalues of the problem* P, *respectively in the segment* Γ_i (γ_i, $\gamma_i^{(n)}$ *are roots of the exact and approximate dispersion equations, respectively,* $i \leq m$, $m \geq 1$). *Then* $\left|\gamma_i^{(n)} - \gamma_i\right| \to 0$ *as* $n \to \infty$.

Proof. Consider functions

$$\Phi(\gamma) = g(\gamma) - \alpha F(\gamma; \mathbf{u}),\ \Phi_n(\gamma) = g(\gamma) - \alpha F(\gamma; \mathbf{u}^n).$$

Then, using estimation (40) and formulas (51)–(53), we find

$$|\Phi(\gamma) - \Phi_n(\gamma)| = \alpha |F(\gamma; \mathbf{u}) - F(\gamma; \mathbf{u}^n)| \leq \\ \leq \alpha \widetilde{C} \|\mathbf{u} - \mathbf{u}^n\| \leq \alpha \widetilde{C} \frac{q^n}{1-q} \|\mathbf{h}\|,$$

where the constant $\widetilde{C}$ does not depend on n and all other values are defined above.

We have

$$\max_{\gamma \in \Gamma} |\Phi(\gamma) - \Phi_n(\gamma)| \leq \alpha \frac{Q^n}{1-Q} C_*, \tag{60}$$

where $C_* = \max_{\gamma \in \Gamma} \left\{ \|\mathbf{h}(\gamma)\| \widetilde{C}(\gamma) \right\}$, $Q = \max_{\gamma \in \Gamma} \left(3 r_*^2(\gamma) \|\mathrm{N}(\gamma)\| \right)$ and $Q < 1$.

Under theorems 4 and 5 conditions the solutions γ_i and $\gamma_i^{(n)}$ of the exact and approximate dispersion equations $\Phi(\gamma) = 0$ and $\Phi_n(\gamma) = 0$ $(n \geq 0)$ exist. When proving theorems 4 and 5 is obtained that the continuous functions $\Phi(\gamma)$, $\Phi_n(\gamma)$ change its signs on the extremities of the interval Γ_i. Then, estimation (60) implies the proof of the theorem.

§12. Numerical Method

The numerical method for calculating approximate eigenvectors and approximate eigenfunctions of the nonlinear boundary problem P is implemented in the following way.

Let us introduce a grid

$$\rho_j = jH_0, \quad j = \overline{0, N-1},$$

where $H_0 = R/N$ in the segment $[0, R]$. All the integrals on the segment $[0, R]$ are calculated by method of rectangles with the nodes $\rho_j^* = jH_0 + H_0/2$. The function $\mathbf{u}^n$ is considered as a mesh function, which is set at the nodes ρ_j^* . To be more precise, $\mathbf{u}^n(\rho) = \mathbf{u}^n(\rho_j^*)$ for $\rho \in \left(\rho_j^* - H_0/2, \rho_j^* + H_0/2\right)$.

Let us introduce a grid $\gamma_{ij} = \sqrt{\lambda_{1i}} + jh_i$, $j = \overline{0, N_i - 1}$, where $h_i = \left(\sqrt{\lambda_{2i}} - \delta_i - \sqrt{\lambda_{1i}}\right)/N_i$ (the step h is sufficiently small) in the segment Γ_i. Then, the values $\Delta(\gamma_{ij})$ are calculated and the segments of signs reversal of $\Delta(\gamma_{ij})$ are defined. In other words, the segments $[\gamma_{ij}, \gamma_{i,j+1}]$ such that $\Delta(\gamma_{ij})\Delta(\gamma_{i,j+1}) < 0$ are defined. In the each of these segments the value of the localized root of equation $\Delta(\gamma) = 0$ is refined by the dichotomy method. Thus the approximate eigenvalues $\widetilde{\gamma}_i^{(n)}$ can be made arbitrary close to the exact roots $\gamma_i^{(n)}$ by means of choosing the steps H_0 and h_i.

Iteration process (58) of solving integral equations system (24) (with fixed γ) begins with the initial approximation $\mathbf{u}^0(s) = (0, 0)^T$ and finishes when the estimation $\max\limits_{0 \leq j \leq N-1} \left|\mathbf{u}^{n+1}(\rho_j^*) - \mathbf{u}^n(\rho_j^*)\right| < \delta$ is fulfilled for certain sufficiently small $\delta > 0$.

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